Hypatia cones reference

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Abstract

We give mathematical descriptions of some cone barriers and oracles for Hypatia's predefined cones. This document is a work in progress. It is not complete, up-to-date, or notationally consistent with our code implementations (see Hypatia's cones folder). It has not been thoroughly checked for mistakes and typographical errors, and some results may be missing sources. Note that our code implementations of most cone oracles are tested using automatic differentiation (see test/runconetests.jl).

Contents

1 Notation

cl	closure of a set
$\mathbb{R}$, $\mathbb{R}_{\geq}$, $\mathbb{R}_{>}$, $\mathbb{R}_{<}$	the reals, nonnegative reals, positive reals, negative reals
$\mathbb{R}^d$, $\mathbb{R}^{d_1 \times d_2}$	the d-dimensional real vectors and d_1-by-d_2-dimensional real matrices
$\mathbb{S}^d$, $\mathbb{S}^d_{\succeq}$, $\mathbb{S}^d_{\succ}$	symmetric, positive semidefinite, positive definite matrices of side d
$\llbracket d \rrbracket$	$\{1, 2, \dots, d\}$
Diag	the diagonal matrix of a given vector
diag	the diagonal vector of a given square matrix
$I(d)$	identity matrix in $\mathbb{R}^{d \times d}$
e_i	ith unit vector
$\lVert \cdot \rVert_p$	ℓ_p-norm (for $p \geq 1$) of a vector
det	determinant of a symmetric matrix
tr	matrix trace
$\mathrm{sd}(d)$	equals $\frac{d(d+1)}{2}$
$\mathrm{vec} : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^{d_1 d_2}$	maps matrices to vectors by stacking columns
$\mathrm{mat}_{d_1, d_2} : \mathbb{R}^{d_1 d_2} \to \mathbb{R}^{d_1 d_2}$	inverse of vec
$\mathrm{vec} : \mathbb{S}^d \to \mathbb{R}^{\mathrm{sd}(d)}$	maps matrices to vectors by stacking columns of the upper triangle, scales off-diagonals by $\sqrt{2}$
$\mathrm{mat} : \mathbb{R}^{\mathrm{sd}(d)} \to \mathbb{S}^d$	inverse of vec
$\sigma_i(\cdot)$	ith largest singular value of a matrix
$W_{i,j}$, $W \in \mathbb{R}^{d_1 \times d_2}$	component in row i, column j of W
$w_{\overline{i,j}}$, $w \in \mathbb{R}^d$	component of w corresponding to (possibly scaled) element $\mathrm{mat}(w)_{i,j}$
$\delta(i,j)$ or $\delta(i,j,k)$	Kronecker delta, takes value one if all inputs are equal and zero otherwise
$\rho(i,j)$	takes value 1 if $i = j$ and value $\sqrt{2}$ otherwise
symm, skron, gkron, akron, mkron, sdot, sdotkron	matrix operators defined in Appendix A
∇f	the gradient vector of f
$\nabla_{w_i} f$	component of the gradient corresponding to w_i
$\nabla^2_{w_i, w_j} f$	component of the Hessian corresponding to w_i and w_j
$\nabla^3_{w_i, w_j, w_k} f$	component of the third order derivative corresponding to w_i, w_j, and w_k
$\nabla^{-2}_{w_i, w_j} f$	component of the inverse Hessian corresponding to w_i and w_j

2 Cones and oracles

Hypatia's cone interface allows the user to specify any primitive proper cone $\mathcal{K} \subseteq \mathbb{R}^q$ by defining a small list of oracles: an initial interior point t, a feasibility check, and gradients and Hessians of a *logarithmically* *homogeneous self-concordant barrier* (LHSCB) function f (with parameter ν) for $\mathcal{K}$. Providing additional oracles is optional; for many cones we omit description of some additional oracles in this reference.

The central point of a primitive proper cone $\mathcal{K}$ with LHSCB f satisfies $t \in \operatorname{int} \mathcal{K} \cap \operatorname{int} \mathcal{K}^*$ and $t = -\nabla f(t)$ [Dahl and Andersen, 2022]. Alternatively, it is the unique solution to the strictly convex problem:

$$t = \arg\min_{s \in \operatorname{int} \mathcal{K}} \left(f(s) + \tfrac{1}{2} \|s\|^2 \right). \tag{1}$$

For some cones, (1) does not have a simple closed form solution. In this case, if $\mathcal{K}$ is parametrized only by its dimension, the central point depends only on the dimension, so we find an approximate central point by interpolating using a nonlinear fit on a range of solutions obtained numerically offline. If we cannot find an approximate central point, we use an initial interior point that may not be close to a central point.

2.1 Nonnegative cone

The self-dual nonnegative cone is:

$$\mathcal{K}_{\geq} = \mathcal{K}_{\geq}^* = \mathbb{R}_{\geq}. \tag{2}$$

For the LHSCB [Nesterov and Todd, 1997, section 2.1]:

$$f(w) = -\log(w) \tag{3}$$

of $\mathcal{K}_{\geq}$, $\nu = 1$, and $t = e$ is a central point. Then:

$$\nabla_w f = -w^{-1}, \tag{4a}$$
$$\nabla^2_{w,w} f = w^{-2}, \tag{4b}$$
$$\nabla^3_{w,w,w} f = -2w^{-3}. \tag{4c}$$

Hypatia uses non-primitive d dimensional nonnegative cones for efficiency.

2.2 Positive semidefinite cone

The self-dual positive semidefinite cone is:

$$\mathcal{K}_{\succeq} = \mathcal{K}_{\succeq}^* = \left\{ w \in \mathbb{R}^{\operatorname{sd}(d)} : W \in \mathbb{S}^d_{\succeq} \right\}, \tag{5}$$

where $W = \operatorname{mat}(w)$. For the LHSCB [Nesterov and Todd, 1997, section 2.2]:

$$f(w) = -\operatorname{logdet}(W) \tag{6}$$

of $\mathcal{K}_{\succeq}$, $\nu = d$, and $t = \operatorname{vec}(I(d))$ is a central point. Then:

$$\nabla_{w_{\overline{i,j}}} f = -\operatorname{vec}(W^{-1})_{\overline{i,j}}, \tag{7a}$$
$$\nabla^2_{w_{\overline{i,j}}, w_{\overline{k,l}}} f = \operatorname{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}), \tag{7b}$$
$$\nabla^3_{w_{\overline{i,j}}, w_{\overline{k,l}}, w_{\overline{m,n}}} f = -\operatorname{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1}). \tag{7c}$$

2.3 Doubly nonnegative cone

The doubly nonnegative cone and its dual cone are:

$$\mathcal{K}_{\mathrm{DNN}} = \left\{ w \in \mathbb{R}^{\operatorname{sd}(d)} : W \in \mathbb{S}^d_{\succeq}, W_{i,j} \in \mathbb{R}_{\geq}, \forall i,j \in [\![d]\!] \right\}, \tag{8a}$$
$$\mathcal{K}^*_{\mathrm{DNN}} = \left\{ w \in \mathbb{R}^{\operatorname{sd}(d)} : \exists w_1 \in \mathcal{K}_{\mathbb{S}_{\succeq}}, w_2 \in \mathbb{R}^{\operatorname{sd}(d)}_{\geq}, w = w_1 + w_2 \right\}, \tag{8b}$$

where $W = \mathrm{mat}(w)$. For the LHSCB (that is a sum of $\mathcal{K}_{\mathbb{R}_{\geq}}$ and $\mathcal{K}_{\mathbb{S}_{\succeq}}$ barriers):

$$f(w) = -\operatorname{logdet}(W) - \textstyle\sum_{j\in[\![d]\!],i\in[\![j-1]\!]} \log\left(w_{\overline{i,j}}\right) \tag{9}$$

of $\mathcal{K}_{\mathrm{DNN}}$, $\nu = \mathrm{sd}(d)$, and a central point t is found by solving the polynomial system:

$$a + \tfrac{(d-2)b}{\sqrt{2}} = a\big(a^2 + \tfrac{(n-2)ab}{\sqrt{2}} - \tfrac{(n-1)b^2}{2}\big), \tag{10a}$$

$$-b^2 + a^2 + \tfrac{(n-2)ab}{\sqrt{2}} - \tfrac{(n-1)b^2}{2} = b^2\big(a^2 + \tfrac{(d-2)ab}{\sqrt{2}} - \tfrac{(d-1)b^2}{2}\big), \tag{10b}$$

for a and b and then letting $w_{\overline{i,i}} = a, \forall i \in [\![d]\!]$ and $w_{\overline{i,j}} = b, \forall i > j$. The values of a and b are functions of the roots of the polynomial $d^2x^6 - \big(2d^2+8\big)x^4 + \big(d^2+d+7\big)d^2 - d - 1$. Then:

$$\nabla_{w_{\overline{i,j}}} f = -\operatorname{vec}(W^{-1})_{\overline{i,j}} - \textstyle\sum_{j\in[\![d]\!],i\in[\![j-1]\!]} w_{\overline{i,j}}^{-1}, \tag{11a}$$

$$\nabla^2_{w_{\overline{i,j}},w_{\overline{k,l}}} f = \operatorname{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}) + \textstyle\sum_{j\in[\![d]\!],i\in[\![j-1]\!]} w_{\overline{i,j}}^{-2}, \tag{11b}$$

$$\nabla^3_{w_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = -\operatorname{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1}) - 2\textstyle\sum_{j\in[\![d]\!],i\in[\![j-1]\!]} w_{\overline{i,j}}^{-3}. \tag{11c}$$

2.4 Sparse positive semidefinite cone

Suppose $\mathcal{S} = ((i_l, j_l))_{l\in[\![d_1]\!]}$ is a collection of row-column index pairs defining the lower triangle sparsity pattern of a symmetric matrix of side dimension d_2 (including all d_2 diagonal elements). Note $d_2 \leq d_1 \leq \mathrm{sd}(d_2)$. Let $\mathrm{mat}_{\mathcal{S}} : \mathbb{R}^{d_1} \to \mathbb{S}^{d_2}$ be the linear operator satisfying for all $i \in [\![d_2]\!], j \in [\![i]\!]$:

$$(\mathrm{mat}_{\mathcal{S}}(w))_{i,j} = \begin{cases} w_l & \text{if } i = i_l = j = j_l, \\ \frac{w_l}{\sqrt{2}} & \text{if } i = i_l \neq j = j_l, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

and for convenience let $W = \mathrm{mat}_{\mathcal{S}}(w)$. The sparse PSD cone and its dual cone, that of PSD-completable matrices, are:

$$\mathcal{K}_{\mathrm{sPSD}(\mathcal{S})} = \big\{w \in \mathbb{R}^{d_1} : W \in \mathbb{S}^{d_2}_{\succeq}\big\}, \tag{13a}$$

$$\mathcal{K}^*_{\mathrm{sPSD}(\mathcal{S})} = \big\{w \in \mathbb{R}^{d_1} : \exists v \in \mathbb{R}^{\mathrm{sd}(d_2)-d_1}, W + \mathrm{mat}_{\bar{\mathcal{S}}}(v) \in \mathbb{S}^{d_2}_{\succeq}\big\}, \tag{13b}$$

where $\bar{\mathcal{S}}$ is the lower triangle inverse sparsity pattern of $\mathcal{S}$ (with dimension $\mathrm{sd}(d_2) - d_1$). For the LHSCB [Nesterov and Nemirovskii, 1994, Proposition 5.1.1]:

$$f(w) = -\operatorname{logdet}(W) \tag{14}$$

of $\mathcal{K}_{\mathrm{sPSD}(\mathcal{S})}$, $\nu = d_2$ and $t_l = \delta(i_l, j_l), \forall l \in [\![d_1]\!]$ is a central point. Then:

$$\nabla_{w_{\overline{i,j}}} f = -\operatorname{vec}(W^{-1})_{\overline{i,j}}, \tag{15a}$$

$$\nabla^2_{w_{\overline{i,j}},w_{\overline{k,l}}} f = \operatorname{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}), \tag{15b}$$

$$\nabla^3_{w_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = -\operatorname{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1}). \tag{15c}$$

2.5 Linear matrix inequality cone

The linear matrix inequality cone, parametrized by d_1 symmetric matrices $A_i \in \mathbb{S}^{d_2}, \forall i \in [\![d_1]\!]$, and its dual cone are:

$$\mathcal{K}_{\mathrm{LMI}(A)} = \big\{w \in \mathbb{R}^{d_1} : \textstyle\sum_{i\in[\![d_1]\!]} w_i A_i \in \mathbb{S}^{d_2}_{\succeq}\big\}, \tag{16a}$$

$$\mathcal{K}^*_{\mathrm{LMI}(A)} = \{ w \in \mathbb{R}^{d_1} : \exists Z \in \mathbb{S}^{d_2}_{\succeq}, \mathrm{tr}(A_i' Z) = w_i, \forall i \in [\![d_1]\!] \}. \tag{16b}$$

For the LHSCB [Nesterov and Nemirovskii, 1994, Proposition 5.1.1]:

$$f(w) = -\operatorname{logdet}\left(\textstyle\sum_{i \in [\![d_1]\!]} w_i A_i \right) \tag{17}$$

of $\mathcal{K}_{\mathrm{LMI}(A)}$, $\nu = d_2$, and $t = e_1$ is a central point if $A_1 = I(d_2)$ (Hypatia assumes $A_1 \succ 0$, without loss of modeling generality). Let $Z = \sum_{i \in [\![d]\!]} w_i A_i$, then:

$$\begin{aligned}
\nabla_{w_i} f &= -\operatorname{tr}(Z^{-1} A_i), && (18a)\\
\nabla^2_{w_i, w_j} f &= \operatorname{tr}(Z^{-1} A_i Z^{-1} A_j), && (18b)\\
\nabla^3_{w_i, w_j, w_k} f &= -2 \operatorname{tr}(Z^{-1} A_i Z^{-1} A_j Z^{-1} A_k). && (18c)
\end{aligned}$$

2.6 Infinity norm cone

The ℓ_∞ norm cone and its dual cone, the ℓ_1 norm cone, are:

$$\begin{aligned}
\mathcal{K}_{\ell_\infty} &= \{ (u, w) \in \mathbb{R}_{\geq} \times \mathbb{R}^d : u \geq \|w\|_\infty \}, && (19a)\\
\mathcal{K}^*_{\ell_\infty} &= \{ (u, w) \in \mathbb{R}_{\geq} \times \mathbb{R}^d : u \geq \|w\|_1 \}. && (19b)
\end{aligned}$$

For the LHSCB [Güler, 1996, section 7.5]:

$$f(u, w) = (d-1)\log(u) - \textstyle\sum_{i \in [\![d]\!]} \log\left(u^2 - w_i^2\right) \tag{20}$$

of $\mathcal{K}_{\ell_\infty}$, $\nu = 1 + d$, and $t = \sqrt{\nu} e_1$ is a central point. Let $z_i = u^2 - w_i^2, \forall i \in [\![d]\!]$, $\tau_i = u^2 + w_i^2, \forall i \in [\![d]\!]$, and $\sigma = \frac{1-d}{u^2} + \sum_{i \in [\![d]\!]} \frac{2}{\tau_i}$, then:

$$\begin{aligned}
\nabla_u f &= \tfrac{d-1}{u} - \textstyle\sum_{i \in [\![d]\!]} \tfrac{2u}{z_i}, && (21a)\\
\nabla_{w_i} f &= \tfrac{2 w_i}{z_i}, && (21b)\\
\nabla^2_{u,u} f &= -\tfrac{d-1}{u^2} + \textstyle\sum_{i \in [\![d]\!]} \tfrac{2\tau_i}{z_i^2}, && (21c)\\
\nabla^2_{u, w_i} f &= \tfrac{-4 u w_i}{z_i^2}, && (21d)\\
\nabla^2_{w_i, w_j} f &= \delta(i,j) \tfrac{2\tau_i}{z_i^2}, && (21e)\\
\nabla^3_{u,u,u} f &= \tfrac{2(d-1)}{u^3} + \textstyle\sum_{i \in [\![d]\!]} \tfrac{4u}{z_i^3}\left(3 z_i - 4u^2\right), && (21f)\\
\nabla^3_{u,u,w_i} f &= \tfrac{4 w_i}{z_i^3}\left(4u^2 - z_i\right), && (21g)\\
\nabla^3_{u, w_i, w_j} f &= -\delta(i,j) \tfrac{4u}{z_i^3}\left(z_i + 4 w_i^2\right), && (21h)\\
\nabla^3_{w_i, w_j, w_k} f &= \delta(i,j,k) \tfrac{4 w_i}{z_i^3}\left(3 z_i + 4 w_i^2\right), && (21i)\\
\nabla^{-2}_{u,u} f &= \tfrac{1}{\sigma}, && (21j)\\
\nabla^{-2}_{u, w_i} f &= \tfrac{2 u w_i}{\sigma \tau_i}, && (21k)\\
\nabla^{-2}_{w_i, w_j} f &= \tfrac{4 u^2 w_i w_j}{\sigma \tau_i \tau_j} + \delta(i,j) \tfrac{z_i^2}{2\tau_i}. && (21l)
\end{aligned}$$

Note that Hypatia implements additional oracles that use the arrowhead structure of the Hessian.

2.7 Euclidean norm cone

The self-dual Euclidean norm cone (second-order cone) is:

$$\mathcal{K}_{\ell_2} = \mathcal{K}^*_{\ell_2} = \{ (u, w) \in \mathbb{R}_{\geq} \times \mathbb{R}^d : u \geq \|w\| \}. \tag{22}$$

For the LHSCB [Nesterov and Todd, 1997, section 2.3]:

$$f(u,w) = -\log\left(u^2 - \|w\|^2\right) \tag{23}$$

of $\mathcal{K}_{\ell_2}$, $\nu = 2$, and $t = \sqrt{\nu} e_1$ is a central point. Let $z = u^2 - \|w\|^2$, and $\tau = u^2 + \|w\|^2$, then:

$$\nabla_u f = \tfrac{-2u}{z}, \tag{24a}$$
$$\nabla_{w_i} f = \tfrac{2w_i}{z}, \tag{24b}$$
$$\nabla^2_{u,u} f = \tfrac{2\tau}{z^2}, \tag{24c}$$
$$\nabla^2_{u,w_i} f = \tfrac{-4uw_i}{z^2}, \tag{24d}$$
$$\nabla^2_{w_i,w_j} f = \tfrac{4w_iw_j}{z^2} + \delta(i,j)\tfrac{2}{z}, \tag{24e}$$
$$\nabla^3_{u,u,u} f = \tfrac{12u}{z^2} - \tfrac{16u^3}{z^3}, \tag{24f}$$
$$\nabla^3_{u,u,w_i} f = \tfrac{16u^2w_i}{z^3} - \tfrac{4w_i}{z^2}, \tag{24g}$$
$$\nabla^3_{u,w_i,w_j} f = \tfrac{-16uw_iw_j}{z^3} - \delta(i,j)\tfrac{4u}{z^2}, \tag{24h}$$
$$\nabla^3_{w_i,w_j,w_k} f = \tfrac{16w_iw_jw_k}{z^3} + \delta(i,j)\tfrac{4w_k}{z^2} + \delta(i,j,k)\tfrac{8w_k}{z^2} \tag{24i}$$
$$\nabla^{-2}_{u,u} f = \tfrac{\tau}{2}, \tag{24j}$$
$$\nabla^{-2}_{u,w_i} f = uw_i, \tag{24k}$$
$$\nabla^{-2}_{w_i,w_j} f = w_iw_j + \delta(i,j)\tfrac{z}{2}. \tag{24l}$$

2.8 Euclidean norm-squared cone

The self-dual Euclidean norm-squared cone (rotated second-order cone) is:

$$\mathcal{K}_{\text{sqr}} = \mathcal{K}^*_{\text{sqr}} = \left\{(u,v,w) \in \mathbb{R}_{\geq} \times \mathbb{R}_{\geq} \times \mathbb{R}^d : 2uv \geq \|w\|^2\right\}. \tag{25}$$

For the LHSCB [Nesterov and Todd, 1997, section 2.3]:

$$f(u,v,w) = -\log\left(2uv - \|w\|^2\right) \tag{26}$$

of $\mathcal{K}_{\text{sqr}}$, $\nu = 2$, and $t = e_1 + e_2$ is a central point. Let $z = uv - \frac{1}{2}\|w\|^2$, then:

$$\nabla_u f = -\tfrac{v}{z}, \tag{27a}$$
$$\nabla_{w_i} f = \tfrac{w_i}{z}, \tag{27b}$$
$$\nabla^2_{u,u} f = \tfrac{v^2}{z^2}, \tag{27c}$$
$$\nabla^2_{u,v} f = \tfrac{uv}{z^2} - \tfrac{1}{z}, \tag{27d}$$
$$\nabla^2_{u,w_i} f = -\tfrac{vw_i}{z^2}, \tag{27e}$$
$$\nabla^2_{w_i,w_j} f = \tfrac{w_iw_j}{z^2} + \delta(i,j)\tfrac{1}{z}, \tag{27f}$$
$$\nabla^3_{u,u,u} f = -\tfrac{2v^3}{z^3}, \tag{27g}$$
$$\nabla^3_{u,u,v} f = \tfrac{2v}{z^2}\left(1 - \tfrac{uv}{z}\right), \tag{27h}$$
$$\nabla^3_{u,u,w_i} f = \tfrac{2v^2w_i}{z^3}, \tag{27i}$$
$$\nabla^3_{u,v,w_i} f = \left(\tfrac{2uv}{z} - 1\right)\tfrac{w_i}{z^2}, \tag{27j}$$
$$\nabla^3_{u,w_i,w_j} f = -\tfrac{2vw_iw_j}{z^3} - \delta(i,j)\tfrac{v}{z^2}, \tag{27k}$$
$$\nabla^3_{w_i,w_j,w_k} f = \tfrac{2w_iw_jw_k}{z^3} + \delta(i,j)\tfrac{w_k}{z^2} + \delta(i,j,k)\tfrac{2w_k}{z^2}, \tag{27l}$$
$$\nabla^{-2}_{u,u} f = u^2, \tag{27m}$$

$$\nabla^{-2}_{u,v} f = uv - z, \tag{27n}$$
$$\nabla^{-2}_{u,w_i} f = uw_i, \tag{27o}$$
$$\nabla^{-2}_{w_i,w_j} f = w_i w_j + \delta(i,j) z. \tag{27p}$$

Note that the barrier and oracles are symmetric with respect to u and v.

2.9 Spectral norm cone

The spectral norm cone and its dual cone, the nuclear norm cone, are:

$$\mathcal{K}_{\ell_{\mathrm{spec}}(d_1,d_2)} = \{(u,w) \in \mathbb{R}_{\geq} \times \mathbb{R}^{d_1 d_2} : u \geq \sigma_1(W)\}, \tag{28a}$$
$$\mathcal{K}^*_{\ell_{\mathrm{spec}}(d_1,d_2)} = \{(u,w) \in \mathbb{R}_{\geq} \times \mathbb{R}^{d_1 d_2} : u \geq \textstyle\sum_{i \in [\![d_1]\!]} \sigma_i(W)\}, \tag{28b}$$

where $W = \mathrm{mat}_{d_1,d_2}(w) \in \mathbb{R}^{d_1 \times d_2}$ and $d_1 \leq d_2$ (nonrestrictive since $\sigma_i(W) = \sigma_i(W'))$. For the LHSCB [Nesterov and Nemirovskii, 1994, section 5.4.6]:

$$f(u,w) = -\log(u) - \mathrm{logdet}\left(uI(d_1) - \tfrac{WW'}{u}\right) = (d_1 - 1)\log(u) - \mathrm{logdet}\left(u^2 I(d_1) - WW'\right) \tag{29}$$

of $\mathcal{K}_{\ell_{\mathrm{spec}}(d_1,d_2)}$, $\nu = 1 + d_1$, and $t = \sqrt{\nu} e_1$ is a central point. Let $Z = u^2 I(d_1) - WW'$, $\Theta = Z^{-1} W$, then:

$$\nabla_u f = -2u \,\mathrm{tr}(Z^{-1}) + \tfrac{d_1 - 1}{u}, \tag{30a}$$
$$\nabla_{w_{\overline{i,j}}} f = 2\Theta_{i,j}, \tag{30b}$$
$$\nabla^2_{u,u} f = 4u^2 \,\mathrm{tr}(Z^{-2}) - 2\,\mathrm{tr}(Z^{-1}) - \tfrac{d_1 - 1}{u^2}, \tag{30c}$$
$$\nabla^2_{u,w_{\overline{i,j}}} f = -4u(Z^{-1}\Theta)_{i,j}, \tag{30d}$$
$$\nabla^2_{w_{\overline{i,j}},w_{\overline{k,l}}} f = 2\,\mathrm{gkron}_{\overline{i,j},\overline{k,l}}\left(Z^{-1}, W'\Theta + I(d_2)\right) + \mathrm{akron}_{\overline{i,j},\overline{k,l}}(\Theta), \tag{30e}$$
$$\nabla^3_{u,u,u} f = 12u \,\mathrm{tr}(Z^{-2}) - 16u^3 \,\mathrm{tr}(Z^{-3}) + 2\tfrac{d_1 - 1}{u^3}, \tag{30f}$$
$$\nabla^3_{u,u,w_{\overline{i,j}}} f = 16u^2 \left(Z^{-2}\Theta\right)_{i,j} - 4\left(Z^{-1}\Theta\right)_{i,j}, \tag{30g}$$
$$\nabla^3_{u,w_{\overline{i,j}},w_{\overline{k,l}}} f = -4u\big(\mathrm{gkron}_{\overline{i,j},\overline{k,l}}\left(Z^{-2}, W'\Theta + I(d_2)\right) + \mathrm{gkron}_{\overline{i,j},\overline{k,l}}\left(Z^{-1}, \Theta'\Theta\right) + \mathrm{akron}_{\overline{i,j},\overline{k,l}}\left(\Theta, Z^{-1}\Theta\right)\big), \tag{30h}$$
$$\nabla^3_{w_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = 2\,\mathrm{gkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}\left(Z^{-1}, \Theta, W'\Theta + I(d_2)\right) + \tfrac{1}{3}\,\mathrm{akron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(\Theta). \tag{30i}$$

2.10 Matrix square cone

The matrix square cone (or Siegel cone) and its dual cone are:

$$\mathcal{K}_{\mathrm{matsqr}(d_1,d_2)} = \{(u,v,w) \in \mathbb{R}^{\mathrm{sd}(d_1)} \times \mathbb{R}_{\geq} \times \mathbb{R}^{d_1 d_2} : U \in \mathbb{S}^{d_1}_{\succeq}, 2vU - WW' \in \mathbb{S}^{d_1}_{\succeq}\}, \tag{31a}$$
$$\mathcal{K}^*_{\mathrm{matsqr}(d_1,d_2)} = \mathrm{cl}\,\{(u,v,w) \in \mathbb{R}^{\mathrm{sd}(d_1)} \times \mathbb{R} \times \mathbb{R}^{d_1 d_2} : U \in \mathbb{S}^{d_1}_{\succ}, 2v > \mathrm{tr}(W'U^{-1}W)\}, \tag{31b}$$

where $U = \mathrm{mat}(u) \in \mathbb{S}^{d_1}$, $W = \mathrm{mat}_{d_1,d_2}(w) \in \mathbb{R}^{d_1 \times d_2}$ and $d_1 \leq d_2$. For the LHSCB [Tunçel and Truong, 2004]:

$$f(u,v,w) = (d_1 - 1)\log(v) - \mathrm{logdet}\left(2vU - WW'\right) \tag{32}$$

of $\mathcal{K}_{\mathrm{matsqr}(d_1,d_2)}$, $\nu = 1 + d_1$, and $t = (\mathrm{vec}(I(d_1)), 1, 0)$ is a central point. Let $U = \mathrm{mat}(u)$, $W = \mathrm{mat}_{d_1,d_2}(w)$, $Z = 2vU - WW'$, $\Theta = Z^{-1}W$, and $\Phi = Z^{-1}U$, then:

$$\nabla_{u_{\overline{i,j}}} f = -2v\,\mathrm{vec}(Z^{-1})_{\overline{i,j}}, \tag{33a}$$
$$\nabla_v f = -2\,\mathrm{tr}(\Phi) + \tfrac{d_1 - 1}{v}, \tag{33b}$$

$$\nabla_{w_{\overline{i,j}}} f = 2\Theta_{i,j}, \tag{33c}$$
$$\nabla^2_{u_{\overline{i,j}},u_{\overline{k,l}}} f = 4v^2 \operatorname{skron}_{\overline{i,j},\overline{k,l}}(Z^{-1}), \tag{33d}$$
$$\nabla^2_{u_{\overline{i,j}},v} f = 2\operatorname{vec}\left(2v\Phi Z^{-1} - Z^{-1}\right)_{\overline{i,j}}, \tag{33e}$$
$$\nabla^2_{u_{\overline{i,j}},w_{\overline{k,l}}} f = -2v \operatorname{mkron}_{\overline{i,j},\overline{k,l}}\left(Z^{-1}, \Theta\right), \tag{33f}$$
$$\nabla^2_{v,v} f = 4\operatorname{tr}(\Phi^2) - \tfrac{d_1 - 1}{v^2}, \tag{33g}$$
$$\nabla^2_{v,w_{\overline{i,j}}} f = -4(\Phi\Theta)_{\overline{i,j}}, \tag{33h}$$
$$\nabla^2_{w_{\overline{i,j}},w_{\overline{k,l}}} f = 2\operatorname{gkron}_{\overline{i,j},\overline{k,l}}\left(Z^{-1}, W'\Theta + I(d_2)\right) + \operatorname{akron}_{\overline{i,j},\overline{k,l}}(\Theta), \tag{33i}$$
$$\nabla^3_{u_{\overline{i,j}},u_{\overline{k,l}},u_{\overline{m,n}}} f = -8v^3 \operatorname{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(Z^{-1}), \tag{33j}$$
$$\nabla^3_{u_{\overline{i,j}},u_{\overline{k,l}},v} f = 8v\big(\operatorname{skron}_{\overline{i,j},\overline{k,l}}(Z^{-1}) - v\operatorname{skron}_{\overline{i,j},\overline{k,l}}(\Phi Z^{-1}, Z^{-1})\big), \tag{33k}$$
$$\nabla^3_{u_{\overline{i,j}},u_{\overline{k,l}},w_{\overline{m,n}}} f = v^2 \operatorname{mkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}\left(Z^{-1}, \Theta, Z^{-1}\right), \tag{33l}$$
$$\nabla^3_{u_{\overline{i,j}},v,v} f = 8\operatorname{vec}\left(\Phi Z^{-1} - 2v\Phi^2 Z^{-1}\right)_{\overline{i,j}}, \tag{33m}$$
$$\nabla^3_{u_{\overline{i,j}},v,w_{\overline{k,l}}} f = 2\big(\operatorname{mkron}_{\overline{i,j},\overline{k,l}}\left(2v\Phi Z^{-1} - Z^{-1}, \Theta\right) + 2v\operatorname{mkron}_{\overline{i,j},\overline{k,l}}\left(Z^{-1}, \Phi\Theta\right)\big), \tag{33n}$$
$$\nabla^3_{u_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = -v\big(\operatorname{mkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}\left(Z^{-1}, \Theta, \Theta\right) + \operatorname{mkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}\left(Z^{-1}, Z^{-1}, W'\Theta + I(d_1)\right)\big), \tag{33o}$$
$$\nabla^3_{v,v,v} f = -16\operatorname{tr}(\Phi^3) + 2\tfrac{d_1 - 1}{v^3}, \tag{33p}$$
$$\nabla^3_{v,v,w_{\overline{i,j}}} f = 16\left(\Phi^2\Theta\right)_{\overline{i,j}}, \tag{33q}$$
$$\nabla^3_{v,w_{\overline{i,j}},w_{\overline{k,l}}} f = -4\big(\operatorname{gkron}_{\overline{i,j},\overline{k,l}}\left(Z^{-1}, W'\Phi\Theta\right) + \operatorname{gkron}_{\overline{i,j},\overline{k,l}}\left(\Phi Z^{-1}, W'\Theta + I(d_2)\right) + \operatorname{akron}_{\overline{i,j},\overline{k,l}}(\Theta, \Phi\Theta)\big), \tag{33r}$$
$$\nabla^3_{w_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = 2\operatorname{gkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}\left(Z^{-1}, \Theta, W'\Theta + I(d_2)\right) + \tfrac{1}{3}\operatorname{akron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(\Theta). \tag{33s}$$

2.11 Generalized power cone

The generalized power cone, parametrized by $\alpha \in \mathbb{R}^{d_1}_{>}$ such that $\sum_{i \in [\![d]\!]} \alpha_i = 1$, and its dual cone are:

$$\mathcal{K}_{\mathrm{gpow}(\alpha)} = \left\{(u, w) \in \mathbb{R}^{d_1}_{+} \times \mathbb{R}^{d_2} : \textstyle\prod_{i \in [\![d_1]\!]} u_i^{\alpha_i} \geq \|w\|\right\}, \tag{34a}$$
$$\mathcal{K}^*_{\mathrm{gpow}(\alpha)} = \left\{(u, w) \in \mathbb{R}^{d_1}_{+} \times \mathbb{R}^{d_2} : \textstyle\prod_{i \in [\![d_1]\!]} \left(\frac{u_i}{\alpha_i}\right)^{\alpha_i} \geq \|w\|\right\}. \tag{34b}$$

For the LHSCB [Roy and Xiao, 2022]:

$$f(u, w) = -\log\left(\textstyle\prod_{i \in [\![d_1]\!]} u_i^{2\alpha_i} - \|w\|^2\right) - \textstyle\sum_{i \in [\![d_1]\!]} (1 - \alpha_i)\log(u_i) \tag{35}$$

of $\mathcal{K}_{\mathrm{gpow}(\alpha)}$, $\nu = d_1 + 1$, and $t = \left((\sqrt{1 + \alpha_i})_{i \in [\![d_1]\!]}, 0\right)$ is a central point. Let $\varphi = \prod_{i \in [\![d_1]\!]} u_i^{2\alpha_i}$, $\tau_i = \frac{2\alpha_i}{u_i}, \forall i \in [\![d_1]\!]$, and $z = \prod_{i \in [\![d_1]\!]} u_i^{2\alpha_i} - \|w\|^2$, then:

$$\nabla_{u_i} f = -\tfrac{\tau_i \varphi}{z} - \tfrac{1 - \alpha_i}{u_i}, \tag{36a}$$
$$\nabla_{w_i} f = \tfrac{2w_i}{z}, \tag{36b}$$
$$\nabla^2_{u_i,u_j} f = \tfrac{\tau_i \tau_j \varphi}{z}\left(\tfrac{\varphi}{z} - 1\right) + \delta(i,j)\left(\tfrac{\tau_i \varphi}{z u_i} + \tfrac{1 - \alpha_i}{u_i^2}\right), \tag{36c}$$
$$\nabla^2_{u_i,w_j} f = -\tfrac{2\tau_i \varphi w_j}{z^2}, \tag{36d}$$
$$\nabla^2_{w_i,w_j} f = \tfrac{4w_i w_j}{z^2} + \delta(i,j)\tfrac{2}{z}, \tag{36e}$$
$$\nabla^3_{u_i,u_j,u_k} f = \tau_i \tau_j \tau_k \tfrac{\varphi}{z}\left(1 - \tfrac{\varphi}{z}\right)\left(\tfrac{2\varphi}{z} - 1\right) + \begin{cases} \frac{3\tau_i^2 \varphi}{uz}\left(1 - \frac{\varphi}{z}\right) - (1 - \alpha_i)\frac{2}{u_i^3} - \frac{2\varphi\tau_i}{z u_i^2} & i = j = k, \\ \frac{\tau_i \tau_k \varphi}{u_j z}\left(1 - \frac{\varphi}{z}\right) & i = j \neq k, \\ 0 & \text{otherwise}, \end{cases} \tag{36f}$$

$$\nabla^3_{u_i,u_j,w_k} f = \tfrac{2\tau_i\tau_j\varphi w_k}{z^2}\big(\tfrac{2\varphi}{z} - 1\big) + \delta(i,j)\tfrac{4\tau_i\varphi^2 w_k}{z^3 u_i}, \tag{36g}$$

$$\nabla^3_{u_i,w_j,w_k} f = \tfrac{-8w_jw_k\tau_i\varphi}{z^3} - \delta(j,k)\tfrac{2}{z^2}\tau_i\varphi, \tag{36h}$$

$$\nabla^3_{w_i,w_j,w_k} f = \tfrac{16w_iw_jw_k}{z^3} + \begin{cases} \tfrac{12u_i}{z^2} & i=j=k, \\ \tfrac{4u_k}{z^2} & i=j\neq k, \\ 0 & \text{otherwise}. \end{cases} \tag{36i}$$

In addition, let:

$$k_1 = \langle \tfrac{\alpha}{\nabla_u f}, \tfrac{\alpha}{u} \rangle, \tag{37a}$$

$$k_2 = \tfrac{1}{2}\big(1 + \tfrac{\|w\|^2}{\varphi}\big) + k_1 \tfrac{\|w\|^2}{z}. \tag{37b}$$

Then:

$$\nabla^{-2}_{u_i,u_j} f = -\delta(i,j)\tfrac{u_i}{\nabla_{u_i} f} - \tfrac{2\|w\|^2}{k_2\zeta}\tfrac{\alpha_i}{\nabla_{u_i} f}\tfrac{\alpha_j}{\nabla_{u_j} f}, \tag{38a}$$

$$\nabla^{-2}_{u_i,w_j} f = -\tfrac{1}{k_2}\tfrac{\alpha_i}{\nabla_{u_i} f} w_j, \tag{38b}$$

$$\nabla^{-2}_{w_i,w_j} f = \delta(i,j)\tfrac{\zeta}{2} - \tfrac{2k_1\varphi + zk_2}{k_2(\varphi + \|w\|^2)} w_i w_j. \tag{38c}$$

2.12 Power mean cone

The power mean cone, parametrized by powers $\alpha \in \mathbb{R}^d_>$ such that $\sum_{i\in[\![d]\!]}\alpha_i = 1$, and its dual cone are:

$$\mathcal{K}_{\text{pow}(\alpha)} = \big\{(u,w) \in \mathbb{R} \times \mathbb{R}^d_\geq : u \leq \textstyle\prod_{i\in[\![d]\!]} w_i^{\alpha_i}\big\}, \tag{39a}$$

$$\mathcal{K}^*_{\text{pow}(\alpha)} = \big\{(u,w) \in \mathbb{R}_\leq \times \mathbb{R}^d_\geq : -u \leq \textstyle\prod_{i\in[\![d]\!]} \big(\tfrac{w_i}{\alpha_i}\big)^{\alpha_i}\big\}. \tag{39b}$$

Since we have not seen a proof of (39b), we provide an outline here that is similar to Chares [2009, Theorem 4.3.1].

Lemma 1. *The dual of $\mathcal{K}_{\text{pow}(\alpha)}$ is given by* (39b)*.*

Proof. Let $(u,w) \in \mathcal{K}_{\text{pow}(\alpha)}$, $\varphi(w) = \prod_{i\in[\![d]\!]} w_i^{\alpha_i}$. For one direction, suppose (p,r) is in the set defined in (39b). Then:

$$\begin{aligned} \langle (u,w),(p,r)\rangle &= up + \langle w,r\rangle && (40a)\\ &\geq -u\varphi(\tfrac{r}{\alpha}) + \langle w,r\rangle && (40b)\\ &\geq -\varphi(w)\varphi(\tfrac{r}{\alpha}) + \langle w,r\rangle && (40c)\\ &= -\textstyle\prod_{i\in[\![d]\!]}(\tfrac{w_ir_i}{\alpha_i})^{\alpha_i} + \sum_{i\in[\![d]\!]}\alpha_i\tfrac{r_i}{\alpha_i}w_i && (40d)\\ &\geq 0. && (40e) \end{aligned}$$

The last inequality follows from a weighted version of the arithmetic-mean-geometric-mean inequality [Chares, 2009, Page 128]. For the other direction, suppose $\langle (u,w),(p,r)\rangle \geq 0$. Suppose (u,w) is such that $u = 0$. Then we are left with the condition $\langle w,r\rangle \geq 0$ for all $w \in \mathbb{R}^d_\geq$, which means that $r \in \mathbb{R}^d_\geq$ by self-duality of $\mathbb{R}^d_\geq$. Next, suppose with a view to contradict that $p > 0$. Note that if $up + \langle w,r\rangle \geq 0$, it must also hold that $\bar{u}p + \langle w,r\rangle \geq 0$ for an arbitrarily negative $\bar{u} < u$ (which retains the condition $(\bar{u},w) \in \mathcal{K}_{\text{pow}(\alpha)}$). So we can find a sufficiently small $\bar{u}$ such that $\bar{u}p + \langle w,r\rangle < 0$, leading to a contradiction. Hence $p < 0$. Finally, suppose that $u = \prod_{i\in[\![d]\!]} w_i^{\alpha_i}$. Then:

$$\langle (u,w),(p,r)\rangle = -\textstyle\prod_{i\in[\![d]\!]} w_i^{\alpha_i}(-p) + \sum_{i\in[\![d]\!]}\alpha_i w_i \tfrac{r_i}{\alpha_i} \geq 0. \tag{41}$$

If $-p > \varphi(\frac{r}{\alpha})$, the weighted version of the arithmetic-mean-geometric-mean inequality is violated. Hence $-p \leq \varphi(\frac{r}{\alpha})$ and the second direction is proved. □

For the LHSCB [Nesterov, 2018, section 5.4.7]:

$$f(u, w) = -\log\big(\textstyle\prod_{i\in[\![d]\!]} w_i^{\alpha_i} - u\big) - \sum_{i\in[\![d]\!]}\log(w_i) \tag{42}$$

of $\mathcal{K}_{\text{pow}(\alpha)}$, $\nu = 1 + d$, and t is obtained by interpolation. Let $z = \prod_{i\in[\![d]\!]} w_i^{\alpha_i} - u$, $\rho = z + u$, and $\tau_i = \frac{\alpha_i}{w_i z}, \forall i \in [\![d]\!]$, then:

$$\nabla_u f = \tfrac{1}{z}, \tag{43a}$$
$$\nabla_{w_i} f = -\rho\tau_i - \tfrac{1}{w_i}, \tag{43b}$$
$$\nabla^2_{u,u} f = \tfrac{1}{z^2}, \tag{43c}$$
$$\nabla^2_{u,w_i} f = -\tfrac{\rho\tau_i}{z}, \tag{43d}$$
$$\nabla^2_{w_i,w_j} f = \rho\tau_i\tau_j u + \delta(i,j)\big(\tfrac{\rho\tau_i}{w_i} + \tfrac{1}{w_i^2}\big), \tag{43e}$$
$$\nabla^3_{u,u,u} f = \tfrac{2}{z^3}, \tag{43f}$$
$$\nabla^3_{u,u,w_i} f = -\tfrac{2\rho\tau_i}{z^2}, \tag{43g}$$
$$\nabla^3_{u,w_i,w_j} f = \rho\tau_i\tau_j\big(\tfrac{2\rho}{z} - 1\big) + \delta(i,j)\tfrac{\tau_i\rho}{w_i z}, \tag{43h}$$
$$\nabla^3_{w_i,w_j,w_k} f = -u\rho\tau_i\tau_j\tau_k\big(2u + z\big) - \delta(i,j)u\tfrac{\rho\tau_i\tau_j}{w} - \delta(i,j,k)\big(\tfrac{2\rho\tau}{w}\big(u\tau + \tfrac{1}{w}\big) + \tfrac{2}{w_i^3}\big). \tag{43i}$$

In addition, if we let:

$$s_{0,i} = 1 + \alpha_i\rho z^{-1} \qquad \forall i \in [\![d]\!], \tag{44a}$$
$$s_1 = \textstyle\sum_{i\in[\![d]\!]} \frac{\alpha_i^2}{s_{0,i}}, \tag{44b}$$
$$s_2 = 1 - \rho z^{-1} s_1, \tag{44c}$$

we have that:

$$\nabla^{-2}_{u,u} f = \zeta(u, w)^2 + \tfrac{s_1}{s_2} u^2, \tag{45a}$$
$$\nabla^{-2}_{w_i,u} f = \tfrac{\rho}{s_2}\tfrac{\alpha_i w_i}{s_{0,i}}, \tag{45b}$$
$$\nabla^{-2}_{w_i,w_j} f = \delta(i,j)\tfrac{w_i^2}{s_{0,i}} + \tfrac{z^{-1}\rho}{s_2}\tfrac{\alpha_i w_i}{s_{0,i}}\tfrac{\alpha_j w_j}{t_j}. \tag{45c}$$

2.13 Geometric mean cone

In the special case $\alpha = d^{-1}e$, the power mean cone is equivalent to the geometric mean cone $\mathcal{K}_{\text{geo}}$, for which we implement additional oracles.

$$\mathcal{K}_{\text{geo}} = \big\{(u, w) \in \mathbb{R} \times \mathbb{R}^d_{\geq} : u \leq \textstyle\prod_{i\in[\![d]\!]} w_i^{1/d}\big\}, \tag{46a}$$
$$\mathcal{K}^*_{\text{geo}} = \big\{(u, w) \in \mathbb{R}_{\leq} \times \mathbb{R}^d_{\geq} : -u \leq d\textstyle\prod_{i\in[\![d]\!]} w_i^{1/d}\big\}. \tag{46b}$$

For $\mathcal{K}_{\text{geo}}$, we use the central point $t = \big(-\big(\frac{a}{2(d+1)}\big)^{1/2}, \frac{b-d+2}{2a\sqrt{d+1}}e\big)$, where $a = 3d - b + 1$ and $b = \sqrt{5d^2 + 2d + 1}$. For the inverse Hessian:

$$\nabla^{-2}_{u,u} f = z^2 + \tfrac{\rho^2}{d}, \tag{47a}$$
$$\nabla^{-2}_{u,w_i} f = \tfrac{w_i\rho}{d}, \tag{47b}$$
$$\nabla^{-2}_{w_i,w_j} f = \tfrac{1}{dz+\rho}\big(\tfrac{\rho}{d} w_i w_j + \delta(i,j) dz w_i^2\big). \tag{47c}$$

2.14 Root-determinant cone

The root-determinant cone and its dual cone are:

$$\mathcal{K}_{\text{rtdet}} = \big\{(u, w) \in \mathbb{R} \times \mathbb{R}^{\text{sd}(d)} : W \in \mathbb{S}^d_{\succeq}, u \leq (\det(W))^{1/d}\big\}, \tag{48a}$$
$$\mathcal{K}^*_{\text{rtdet}} = \big\{(u, w) \in \mathbb{R}_{\leq} \times \mathbb{R}^{\text{sd}(d)} : W \in \mathbb{S}^d_{\succeq}, -u \leq d(\det(W))^{1/d}\big\}, \tag{48b}$$

where $W = \text{mat}(w)$. For the LHSCB [Coey et al., 2021, Proposition 7.1]:

$$f(u, w) = -\log\big((\det(W))^{1/d} - u\big) - \text{logdet}(W) \tag{49}$$

of $\mathcal{K}_{\text{rtdet}}$, $\nu = (1 + d)$, and $t = \big(-c_2, \big(\frac{d+1-c_1}{2d}\big)c_2 \,\text{vec}(I(d))\big)$ is a central point, where $c_1 = (5d^2 + 2d + 1)^{1/2}$ and $c_2 = \frac{5}{3}\big(\frac{3d+1-c_1}{2(d+1)}\big)^{1/2}$. Let $\alpha = (\det(W))^{1/d}$ and $z = \alpha - u$ and $\sigma = \frac{1}{zd}\alpha$, then:

$$\nabla_u f = \tfrac{1}{z}, \tag{50a}$$
$$\nabla_{w_{\overline{i,j}}} f = -\big(\sigma + 1\big)\,\text{vec}(W^{-1})_{\overline{i,j}}, \tag{50b}$$
$$\nabla^2_{u,u} f = \tfrac{1}{z^2}, \tag{50c}$$
$$\nabla^2_{u,w_{\overline{i,j}}} f = -\tfrac{\sigma}{z}\,\text{vec}(W^{-1})_{\overline{i,j}}, \tag{50d}$$
$$\nabla^2_{w_{\overline{i,j}},w_{\overline{k,l}}} f = -\sigma\big(\tfrac{1}{d_1} - \sigma\big)\,\text{sdot}_{\overline{i,j},\overline{k,l}}(W^{-1}) + (\sigma + 1)\,\text{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}), \tag{50e}$$
$$\nabla^3_{u,u,u} f = \tfrac{2}{z^3}, \tag{50f}$$
$$\nabla^3_{u,u,w_{\overline{i,j}}} f = -\tfrac{2\sigma}{z^2}\,\text{vec}(W^{-1})_{\overline{i,j}}, \tag{50g}$$
$$\nabla^3_{u,w_{\overline{i,j}},w_{\overline{k,l}}} f = \tfrac{\sigma}{z}\big(\big(2\sigma - \tfrac{1}{d}\big)\,\text{sdot}_{\overline{i,j},\overline{k,l}}(W^{-1}) + \text{skron}_{\overline{i,j},\overline{k,l}}(W^{-1})\big), \tag{50h}$$
$$\begin{aligned}\nabla^3_{w_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = \big(&-\sigma\big(\tfrac{1}{d} - \sigma\big)\big(\tfrac{1}{d} - 2\sigma\big)\,\text{sdot}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1}) + \\ &\sigma\big(\tfrac{1}{d} - \sigma\big)\,\text{sdotkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1}) - (\sigma + 1)\,\text{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1})\big),\end{aligned} \tag{50i}$$
$$\nabla^{-2}_{u,u} f = \big(z^2 + \tfrac{\alpha^2}{d}\big), \tag{50j}$$
$$\nabla^{-2}_{u,w_i} f = \tfrac{w\alpha}{d}, \tag{50k}$$
$$\nabla^{-2}_{w_{\overline{i,j}},w_{\overline{k,l}}} f = \tfrac{1}{d(dz+\alpha)}\big(d^2 z\,\text{skron}_{\overline{i,j},\overline{k,l}}(W) + \alpha\,\text{sdot}_{\overline{i,j},\overline{k,l}}(W^{-1})\big). \tag{50l}$$

2.15 Logarithm cone

The logarithm cone and its dual cone are:

$$\mathcal{K}_{\log} = \text{cl}\,\big\{(u, v, w) \in \mathbb{R} \times \mathbb{R}_{>} \times \mathbb{R}^d_{>} : u \leq \textstyle\sum_{i \in [\![d]\!]} v \log\big(\frac{w_i}{v}\big)\big\}, \tag{51a}$$
$$\mathcal{K}^*_{\log} = \text{cl}\,\big\{(u, v, w) \in \mathbb{R}_{<} \times \mathbb{R} \times \mathbb{R}^d_{>} : v \geq \textstyle\sum_{i \in [\![d]\!]} u\big(\log\big(\frac{-w_i}{u}\big) + 1\big)\big\}. \tag{51b}$$

For the LHSCB [Coey et al., 2021, Proposition 6.1]:

$$f(u, v, w) = -\log\big(\textstyle\sum_{i \in [\![d]\!]} v \log\big(\frac{w_i}{v}\big) - u\big) - \log(v) - \sum_{i \in [\![d]\!]} \log(w_i) \tag{52}$$

of $\mathcal{K}_{\log}$, $\nu = d + 2$, and t is obtained by interpolation. Let $z = \sum_{i \in [\![d]\!]} v \log\big(\frac{w_i}{v}\big) - u$, $\tau = \sum_{i \in [\![d]\!]} \log\big(\frac{w_i}{v}\big) - d$, and $\sigma = z + v(1 + d)$, then:

$$\nabla_u f = \tfrac{1}{z}, \tag{53a}$$
$$\nabla_v f = -\tfrac{\tau}{z} - \tfrac{1}{v}, \tag{53b}$$
$$\nabla_{w_i} f = -\tfrac{v}{zw_i} - \tfrac{1}{w_i}, \tag{53c}$$
$$\nabla_{u,u} f = \tfrac{1}{z^2}, \tag{53d}$$

$$\nabla_{u,v} f = -\tfrac{\tau}{z^2}, \tag{53e}$$
$$\nabla_{u,w_i} f = -\tfrac{v}{z^2 w_i}, \tag{53f}$$
$$\nabla_{v,v} f = \tfrac{\tau^2}{z^2} + \tfrac{d}{vz} + \tfrac{1}{v^2}, \tag{53g}$$
$$\nabla_{v,w_i} f = \tfrac{\tau v}{z^2 w} - \tfrac{1}{wz}, \tag{53h}$$
$$\nabla_{w_i,w_j} f = \tfrac{v^2}{z^2 w_i w_j} + \delta(i,j)\big(\tfrac{v}{z w_i^2} + \tfrac{1}{w_i^2}\big), \tag{53i}$$
$$\nabla_{u,u,u} f = \tfrac{2}{z^3}, \tag{53j}$$
$$\nabla_{u,u,v} f = -\tfrac{2\tau}{z^3}, \tag{53k}$$
$$\nabla_{u,u,w_i} f = -\tfrac{2v}{z^3 w_i}, \tag{53l}$$
$$\nabla_{u,v,v} f = \tfrac{2\tau^2}{z^3} + \tfrac{d}{z^2 v}, \tag{53m}$$
$$\nabla_{u,v,w_i} f = \tfrac{2\tau v}{z^3 w_i} - \tfrac{1}{w_i z^2}, \tag{53n}$$
$$\nabla_{u,w_i,w_j} f = \tfrac{2v^2}{z^3 w_i w_j} + \delta(i,j)\tfrac{v}{z^2 w_i^2}, \tag{53o}$$
$$\nabla_{v,v,v} f = -\tfrac{2\tau^3}{z^3} - \tfrac{3\tau d}{z^2 v} - \tfrac{d}{zv^2} - \tfrac{2}{v^3}, \tag{53p}$$
$$\nabla_{v,v,w_i} f = -\tfrac{2\tau^2 v}{z^3 w_i} + \tfrac{2\tau}{z^2 w_i} - \tfrac{d}{w_i z^2}, \tag{53q}$$
$$\nabla_{v,w_i,w_j} f = -\tfrac{2v^2\tau}{z^3 w_i w_j} + \tfrac{2v}{z^2} + \delta(i,j)\big(-\tfrac{\tau v}{z^2 w_i^2} + \tfrac{1}{w_i^2 z}\big), \tag{53r}$$
$$\nabla_{w_i,w_j,w_k} f = -\tfrac{2v^3}{z^3 w_i w_j w_k} + \begin{cases} -\tfrac{3v^2}{z^2 w_i^3} - \tfrac{2v}{z w_i^3} - \tfrac{2}{w_i^3} & i = j = k, \\ -\tfrac{v^2}{z^2 w_i^2 w_k} & i = j \neq k, \\ 0 & \text{otherwise}, \end{cases} \tag{53s}$$
$$\nabla^{-2}_{u,u} f = (z+u)^2 + z(\sigma - v) - \tfrac{dv}{\sigma}(2z+u)^2, \tag{53t}$$
$$\nabla^{-2}_{u,v} f = v^2\sigma^{-1}((z+v)(\tau+d) - dz), \tag{53u}$$
$$\nabla^{-2}_{u,w_i} f = \tfrac{1}{\sigma} vw(2z+u), \tag{53v}$$
$$\nabla^{-2}_{v,v} f = \tfrac{1}{\sigma} v^2(z+v), \tag{53w}$$
$$\nabla^{-2}_{v,w_i} f = \tfrac{1}{\sigma} v^2 w_i, \tag{53x}$$
$$\nabla^{-2}_{w_i,w_j} f = \tfrac{1}{z+v}\big(\tfrac{v^2 w^2}{\sigma} + \delta(i,j) w_i^2 z\big). \tag{53y}$$

2.16 Log-determinant cone

The log-determinant cone and its dual cone are:

$$\mathcal{K}_{\text{logdet}} = \text{cl}\left\{(u,v,w) \in \mathbb{R} \times \mathbb{R}_{>} \times \mathbb{R}^{\text{sd}(d)} : W \in \mathbb{S}^d_{\succ}, u \leq v \,\text{logdet}\left(\tfrac{W}{v}\right)\right\}, \tag{54a}$$
$$\mathcal{K}^*_{\text{logdet}} = \text{cl}\left\{(u,v,w) \in \mathbb{R}_{<} \times \mathbb{R} \times \mathbb{R}^{\text{sd}(d)} : W \in \mathbb{S}^d_{\succ}, v \geq u\big(\text{logdet}\left(\tfrac{-W}{u}\right) + d\big)\right\}, \tag{54b}$$

where $W = \text{mat}(w)$. For the LHSCB [Coey et al., 2021, Proposition 6.1]:

$$f(u,v,w) = -\log\big(v \,\text{logdet}\left(\tfrac{W}{v}\right) - u\big) - \log(v) - \text{logdet}(W) \tag{55}$$

of $\mathcal{K}_{\text{logdet}}$, $\nu = d+2$, and t is obtained by interpolation on d. Let $\alpha = \text{logdet}\left(\frac{W}{v}\right)$, $z = v\alpha - u$, and $\sigma = z + v(d+1)$, then:

$$\nabla_u f = \tfrac{1}{z}, \tag{56a}$$
$$\nabla_v f = \tfrac{d-\alpha}{z} - \tfrac{1}{v}, \tag{56b}$$
$$\nabla_{w_{\overline{i,j}}} f = -\big(\tfrac{v}{z} + 1\big)\,\text{vec}(W^{-1})_{\overline{i,j}}, \tag{56c}$$
$$\nabla^2_{u,u} f = \tfrac{1}{z^2}, \tag{56d}$$

$$\nabla^2_{u,v} f = \tfrac{d-\alpha}{z^2}, \tag{56e}$$
$$\nabla^2_{u,w_{\overline{i,j}}} f = -\tfrac{v}{z^2} \operatorname{vec}(W^{-1})_{\overline{i,j}}, \tag{56f}$$
$$\nabla^2_{v,v} f = \left(\tfrac{\alpha-d}{z}\right)^2 + \tfrac{d}{zv} + \tfrac{1}{v^2}, \tag{56g}$$
$$\nabla^2_{v,w_{\overline{i,j}}} f = \tfrac{1}{z}\left(\tfrac{v(\alpha-d)}{z} - 1\right) \operatorname{vec}(W^{-1})_{\overline{i,j}}, \tag{56h}$$
$$\nabla^2_{w_{\overline{i,j}},w_{\overline{k,l}}} f = \tfrac{v^2}{z^2} \operatorname{sdot}_{\overline{i,j},\overline{k,l}}(W^{-1}) + (1 + \tfrac{v}{z}) \operatorname{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}), \tag{56i}$$
$$\nabla^3_{u,u,u} f = \tfrac{2}{z^3}, \tag{56j}$$
$$\nabla^3_{u,u,v} f = -\tfrac{2}{z^3}(\alpha - d), \tag{56k}$$
$$\nabla^3_{u,u,w_{\overline{i,j}}} f = -\tfrac{2}{z^3} v \operatorname{vec}(W^{-1})_{\overline{i,j}}, \tag{56l}$$
$$\nabla^3_{u,v,v} f = \tfrac{2(\alpha-d)^2}{z^3} + \tfrac{d}{z^2 v}, \tag{56m}$$
$$\nabla^3_{u,v,w_{\overline{i,j}}} f = \tfrac{1}{z}\left(\tfrac{2v(\alpha-d)}{z^2} - \tfrac{1}{z}\right) \operatorname{vec}(W^{-1})_{\overline{i,j}}, \tag{56n}$$
$$\nabla^3_{u,w_{\overline{i,j}},w_{\overline{k,l}}} f = \tfrac{2v^2}{z^3} \operatorname{sdot}_{\overline{i,j},\overline{k,l}}(W^{-1}) + \tfrac{v}{z^2} \operatorname{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}), \tag{56o}$$
$$\nabla^3_{v,v,v} f = \tfrac{d-\alpha}{z}\left(\tfrac{2(\alpha-d)^2}{z^2} + \tfrac{3d}{vz}\right) - \tfrac{d}{v^2 z} - \tfrac{2}{v^3}, \tag{56p}$$
$$\nabla^3_{v,v,w_{\overline{i,j}}} f = \tfrac{1}{z^2}\left(2(\alpha - d)\left(\tfrac{vd-u}{z}\right) - d\right) \operatorname{vec}(W^{-1})_{\overline{i,j}}, \tag{56q}$$
$$\nabla^3_{v,w_{\overline{i,j}},w_{\overline{k,l}}} f = \left(\tfrac{1}{z} - \tfrac{v(\alpha-d)}{z^2}\right)\left(\tfrac{2v}{z} \operatorname{sdot}_{\overline{i,j},\overline{k,l}}(W^{-1}) + \operatorname{skron}_{\overline{i,j},\overline{k,l}}(W^{-1})\right), \tag{56r}$$
$$\nabla_{w_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = -\tfrac{2v^3}{z^3} \operatorname{sdot}_{\overline{i,j},\overline{k,l},\overline{m,n}}\left(W^{-1}\right) - \tfrac{v^2}{z^2} \operatorname{sdotkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1}) - \left(1 + \tfrac{v}{z}\right) \operatorname{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(W^{-1}), \tag{56s}$$
$$\nabla^{-2}_{u,u} f = (z + u)^2 + z(\sigma - v) - \tfrac{dv}{\sigma}(2z + u)^2, \tag{56t}$$
$$\nabla^{-2}_{u,v} f = v^2\sigma^{-1}((\zeta + v)\hat{\varphi} - d\zeta), \tag{56u}$$
$$\nabla^{-2}_{u,w_{\overline{i,j}}} f = \tfrac{1}{\sigma} v w_{\overline{i,j}}(\alpha + z), \tag{56v}$$
$$\nabla^{-2}_{v,v} f = \tfrac{1}{\sigma} v^2(z + v), \tag{56w}$$
$$\nabla^{-2}_{v,w_{\overline{i,j}}} f = \tfrac{1}{\sigma} v^2 w_{\overline{i,j}}, \tag{56x}$$
$$\nabla^{-2}_{w_{\overline{i,j}},w_{\overline{k,l}}} f = \tfrac{1}{z+v}\left(\tfrac{v^2}{\sigma} \operatorname{sdot}_{\overline{i,j},\overline{k,l}}(W) + z \operatorname{skron}_{\overline{i,j},\overline{k,l}}(W)\right). \tag{56y}$$

2.17 Separable spectral function cone

See [Coey et al., 2021, Proposition 6.1] for more details. Suppose $h : \mathbb{R}_> \to \mathbb{R}$ is a convex function. Let V be a Jordan algebra with rank d and let $\mathcal{Q}$ be its cone of squares. h induces a separable spectral function $\varphi : \mathcal{Q} \to \mathbb{R}$, $\varphi(w) = \sum_{i \in [\![d]\!]} h(\lambda_i)$, where $\lambda_i > 0$ is the ith eigenvalue of $w \in \mathcal{Q}$. The epigraph of the conic hull of φ and its dual cone are:

$$\mathcal{K}_{\text{sepspec}} = \operatorname{cl}\left\{(u, v, w) \in \mathbb{R} \times \mathbb{R}_> \times \mathcal{Q} : u \geq v\varphi(w/v)\right\}, \tag{57a}$$
$$\mathcal{K}^*_{\text{sepspec}} = \operatorname{cl}\left\{(u, v, w) \in \mathbb{R}_> \times \mathbb{R} \times \mathcal{R} : v \geq u\varphi^*(w/u)\right\}, \tag{57b}$$

where $\mathcal{R}$ is the natural domain of φ^*.

Suppose $\mathcal{K}_{\text{sepspec}}$ has the LHSCB [Coey et al., 2021, Proposition 6.1]:

$$f(u, v, w) = -\log(u - v\varphi(w/v)) - \log(v) - \operatorname{logdet}(w) \tag{58}$$

with $\nu = 2 + d$. Let P denote the quadratic form on V. We let $\hat{\varphi} = \varphi(w/v)$ and we denote the derivatives of φ at w/v as $\nabla\varphi$, $\nabla^2\varphi$, $\nabla^3\varphi$. We let $\sigma = \hat{\varphi} - \nabla\varphi[w/v]$ and $\zeta = u - v\hat{\varphi}$. Then:

$$\nabla_u f = -\zeta^{-1}, \tag{59a}$$

$$\nabla_v f = \zeta^{-1}\sigma - v^{-1}, \tag{59b}$$
$$\nabla_w f = \zeta^{-1}\nabla\varphi - w^{-1}, \tag{59c}$$
$$\nabla^2_{u,u} f = \zeta^{-2}, \tag{59d}$$
$$\nabla^2_{v,u} f = -\zeta^{-2}\sigma, \tag{59e}$$
$$\nabla^2_{w,u} f = -\zeta^{-2}\nabla\varphi, \tag{59f}$$
$$\nabla^2_{v,v} f = v^{-2} + \zeta^{-2}\sigma^2 + v^{-1}\zeta^{-1}\nabla^2\varphi[w/v, w/v], \tag{59g}$$
$$\nabla^2_{w,v} f = \zeta^{-2}\sigma\nabla\varphi - v^{-1}\zeta^{-1}\nabla^2\varphi[w/v], \tag{59h}$$
$$\nabla^2_{w,w} f = \zeta^{-2}\nabla\varphi(\nabla\varphi)' + v^{-1}\zeta^{-1}\nabla^2\varphi + P(w^{-1}). \tag{59i}$$

2.18 Vector relative entropy cone

The relative entropy cone and its dual cone are:

$$\mathcal{K}_{\text{relent}} = \text{cl}\left\{(u, v, w) \in \mathbb{R} \times \mathbb{R}^d_> \times \mathbb{R}^d_> : u \geq \textstyle\sum_{i \in [\![d]\!]} w_i \log\left(\frac{w_i}{v_i}\right)\right\}, \tag{60a}$$
$$\mathcal{K}^*_{\text{relent}} = \text{cl}\left\{(u, v, w) \in \mathbb{R}_> \times \mathbb{R}^d_> \times \mathbb{R}^d : w_i \geq u\left(\log\left(\frac{u}{v_i}\right) - 1\right), \forall i \in [\![d]\!]\right\}. \tag{60b}$$

For the LHSCB [Karimi and Tunçel, 2020b, Section 1.4], [Karimi and Tunçel, 2020a, Appendix E]:

$$f(u, v, w) = -\log\left(u - \textstyle\sum_{i \in [\![d]\!]} w_i \log\left(\frac{w_i}{v_i}\right)\right) - \textstyle\sum_{i \in [\![d]\!]}\left(\log(v_i) + \log(w_i)\right) \tag{61}$$

of $\mathcal{K}_{\text{relent}}$, $\nu = 1+2d$, and t is obtained by interpolation. Let $z = u - \sum_{i \in [\![d]\!]} w_i \log\left(\frac{w_i}{v_i}\right)$, $\sigma_i = \frac{w_i}{z v_i}, \forall i \in [\![d]\!]$, $\tau_i = -z^{-1}\left(\log\left(\frac{w_i}{v_i}\right) + 1\right), \forall i \in [\![d]\!]$, $\alpha_i = \frac{1}{z + 2w_i}, \forall i \in [\![d]\!]$, $\beta_i = \log\left(\frac{w_i}{v_i}\right), \forall i \in [\![d]\!]$, and $\gamma_i = \sum_{j \in [\![d]\!] \setminus \{i\}} w_j \beta_j, \forall i \in [\![d]\!]$, then:

$$\nabla_u f = -\tfrac{1}{z}, \tag{62a}$$
$$\nabla_{v_i} f = -\sigma_i - \tfrac{1}{v_i}, \tag{62b}$$
$$\nabla_{w_i} f = -\tfrac{1}{w_i} - \tau_i, \tag{62c}$$
$$\nabla^2_{u,u} f = \tfrac{1}{z^2}, \tag{62d}$$
$$\nabla^2_{u,v_i} f = \tfrac{\sigma_i}{z}, \tag{62e}$$
$$\nabla^2_{u,w_i} f = \tfrac{\tau_i}{z}, \tag{62f}$$
$$\nabla^2_{v_i,v_j} f = \sigma_i\sigma_j + \delta(i,j)\left(\tfrac{\sigma_i}{v_i} + \tfrac{1}{v_i^2}\right), \tag{62g}$$
$$\nabla^2_{v_i,w_i} f = \sigma_i\tau_i - \tfrac{1}{z v_i}, \tag{62h}$$
$$\nabla^2_{v_i,w_j} f = \sigma_i\tau_j, \tag{62i}$$
$$\nabla^2_{w_i,w_j} f = \tau_i\tau_j + \delta(i,j)\left(\tfrac{1}{z w_i} + \tfrac{1}{w_i^2}\right), \tag{62j}$$
$$\nabla^3_{u,u,u} f = -\tfrac{2}{z^3}, \tag{62k}$$
$$\nabla^3_{u,u,v_i} f = -\tfrac{2\sigma_i}{z^2}, \tag{62l}$$
$$\nabla^3_{u,u,w_i} f = -\tfrac{2\tau_i}{z^2}, \tag{62m}$$
$$\nabla^3_{u,v_i,v_j} f = -\tfrac{2\sigma_i\sigma_j}{z} - \delta(i,j)\tfrac{\sigma_i}{v_i z}, \tag{62n}$$
$$\nabla^3_{u,v_i,w_j} f = -\tfrac{2\sigma_i\tau_j}{z} + \delta(i,j)\tfrac{1}{v_i z^2}, \tag{62o}$$
$$\nabla^3_{u,w_i,w_j} f = -\tfrac{2\tau_i\tau_j}{z} - \delta(i,j)\tfrac{1}{w_i z^2}, \tag{62p}$$
$$\nabla^3_{v_i,v_j,v_k} f = -2\sigma_i\sigma_j\sigma_k + \begin{cases} -\frac{3\sigma_i^2}{v_i} - 2\frac{\sigma_i}{v_i^2} - \frac{2}{v_i^3} & i = j = k, \\ -\frac{\sigma_i\sigma_k}{v_1} & i = j \neq k, \\ 0 & \text{otherwise}, \end{cases} \tag{62q}$$

$$\nabla^3_{v_i,v_j,w_k} f = -2\sigma_i\sigma_j\tau_k + \begin{cases} \frac{1}{v_i^2 z} - \frac{\tau_i\sigma_i}{v_i} + \frac{2\sigma_i}{v_i z} & i = j = k, \\ -\frac{\tau_i\sigma_i}{v_i} & i = j \neq k, \\ \frac{\sigma_i}{v_j z} & i \neq j = k, \\ 0 & \text{otherwise}, \end{cases} \tag{62r}$$

$$\nabla^3_{v_i,w_j,w_k} f = -2\sigma_i\tau_j\tau_k + \begin{cases} 2\frac{\tau_k}{v_i z} - \frac{1}{v_i z^2} & i = j = k, \\ \frac{\tau_k}{v_i z} & i = j \neq k, \\ -\frac{\sigma_i}{w_j z} & i \neq j = k, \\ 0 & \text{otherwise}, \end{cases} \tag{62s}$$

$$\nabla^3_{w_i,w_j,w_k} f = -2\tau_i\tau_j\tau_k + \begin{cases} -\frac{3\tau_k}{w_i z} - \frac{1}{w_i^2 z} - \frac{2}{w_i^3} & i = j = k, \\ -\frac{\tau_k}{w_i z} & i = j \neq k, \\ 0 & \text{otherwise}, \end{cases} \tag{62t}$$

$$\nabla^{-2}_{u,u} f = z^2 - \textstyle\sum_{i \in [\![d]\!]} \alpha_i w_i (w_i\beta_i - z + \tau_i z(z + z\beta_i + w_i\beta_i)), \tag{62u}$$

$$\nabla^{-2}_{u,v_i} f = -v_i w_i \alpha_i (u - \gamma_i - 2w_i\beta_i), \tag{62v}$$

$$\nabla^{-2}_{u,w_i} f = -w_i^2 \alpha_i (\beta_i z + u - \gamma_i), \tag{62w}$$

$$\nabla^{-2}_{v_i,v_j} f = \delta(i,j) v_i^2 \alpha_i (z + w_i), \tag{62x}$$

$$\nabla^{-2}_{v_i,w_j} f = \delta(i,j) v_i w_i^2 \alpha_i, \tag{62y}$$

$$\nabla^{-2}_{w_i,w_j} f = \delta(i,j) w_i^2 \alpha_i (z + w_i). \tag{62z}$$

Note that Hypatia implements additional oracles that use the block arrowhead structure of the inverse Hessian.

2.19 Matrix relative entropy cone

The matrix relative entropy cone is:

$$\mathcal{K}_{\mathrm{matrelent}} = \mathrm{cl}\left\{(u,v,w) \in \mathbb{R} \times \mathbb{R}^{\mathrm{sd}(d)} \times \mathbb{R}^{\mathrm{sd}(d)} : V, W \in \mathbb{S}_{\succ}, u \geq \mathrm{tr}(W\log(W)) - \mathrm{tr}(W\log(V))\right\}, \tag{63a}$$

where $V = \mathrm{mat}(v)$ and $W = \mathrm{mat}(w)$. We are not aware of a closed-form expression for the dual cone.

For the LHSCB:

$$f(u,v,w) = -\operatorname{logdet}(V) - \operatorname{logdet}(W) - \log\left(u - \mathrm{tr}(W\log(W)) + \mathrm{tr}(W\log(V))\right) \tag{64}$$

of $\mathcal{K}_{\mathrm{matrelent}}$, $\nu = 1 + 2d$, and t is obtained by interpolation.

Let $z = u - \mathrm{tr}(W\log(W)) + \mathrm{tr}(W\log(V))$. We first give the derivatives of f in terms of derivatives of z:

$$\nabla_u f = -\tfrac{1}{z}, \tag{65}$$

$$\nabla_{v_{\overline{i,j}}} f = -\tfrac{1}{z}\tfrac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}} - \mathrm{vec}(V^{-1})_{i,j}, \tag{66}$$

$$\nabla_{w_{\overline{i,j}}} f = -\tfrac{1}{z}\tfrac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}} - \mathrm{vec}(W^{-1})_{i,j}, \tag{67}$$

$$\nabla^2_{u,u} f = \tfrac{1}{z^2}, \tag{68}$$

$$\nabla^2_{u,v_{\overline{i,j}}} f = \tfrac{1}{z^2}\tfrac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}, \tag{69}$$

$$\nabla^2_{u,w_{\overline{i,j}}} f = \tfrac{1}{z^2}\tfrac{\mathrm{d}z}{\mathrm{d}w_{\overline{i,j}}}, \tag{70}$$

$$\nabla^2_{v_{\overline{i,j}},v_{\overline{k,l}}} f = \tfrac{1}{z^2}\tfrac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}\tfrac{\mathrm{d}z}{\mathrm{d}v_{\overline{k,l}}} - \tfrac{1}{z}\tfrac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}v_{\overline{k,l}}} + \mathrm{skron}_{\overline{i,j},\overline{k,l}}(V^{-1}), \tag{71}$$

$$\nabla^2_{v_{\overline{i,j}},w_{\overline{k,l}}} f = \tfrac{1}{z^2}\tfrac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}\tfrac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}} - \tfrac{1}{z}\tfrac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}w_{\overline{k,l}}}, \tag{72}$$

$$\nabla^2_{w_{\overline{i,j}},w_{\overline{k,l}}} f = \frac{1}{z^2}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}} - \frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}w_{\overline{i,j}}\mathrm{d}w_{\overline{k,l}}} + \mathrm{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}), \tag{73}$$

$$\nabla^3_{u,u,u} f = -\frac{2}{z^3}, \tag{74}$$

$$\nabla^3_{u,u,v_{\overline{i,j}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}, \tag{75}$$

$$\nabla^3_{u,u,w_{\overline{i,j}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{i,j}}}, \tag{76}$$

$$\nabla^3_{u,v_{\overline{i,j}},v_{\overline{k,l}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{k,l}}} + \frac{1}{z^2}\frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}v_{\overline{k,l}}}, \tag{77}$$

$$\nabla^3_{u,v_{\overline{i,j}},w_{\overline{k,l}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}} + \frac{1}{z^2}\frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}w_{\overline{k,l}}}, \tag{78}$$

$$\nabla^3_{u,w_{\overline{i,j}},w_{\overline{k,l}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}} + \frac{1}{z^2}\frac{\mathrm{d}^2 z}{\mathrm{d}w_{\overline{i,j}}\mathrm{d}w_{\overline{k,l}}} + \mathrm{skron}_{\overline{i,j},\overline{k,l}}(W^{-1}), \tag{79}$$

$$\nabla^3_{v_{\overline{i,j}},v_{\overline{k,l}},v_{\overline{m,n}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{k,l}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{m,n}}} + \frac{1}{z^2}\Big(\frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}v_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{k,l}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{k,l}}\mathrm{d}v_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}v_{\overline{k,l}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{m,n}}}\Big) - \tag{80}$$

$$\frac{1}{z}\frac{\mathrm{d}^3 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}v_{\overline{k,l}}\mathrm{d}v_{\overline{m,n}}} - \mathrm{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(V^{-1}), \tag{81}$$

$$\nabla^3_{v_{\overline{i,j}},v_{\overline{k,l}},w_{\overline{m,n}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{k,l}}} + \frac{1}{z^2}\Big(\frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}w_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{k,l}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}w_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}v_{\overline{k,l}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{m,n}}}\Big) - \tag{82}$$

$$\frac{1}{z}\frac{\mathrm{d}^3 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}v_{\overline{k,l}}\mathrm{d}w_{\overline{m,n}}}, \tag{83}$$

$$\nabla^3_{v_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{m,n}}} + \frac{1}{z^2}\Big(\frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}w_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}w_{\overline{k,l}}\mathrm{d}w_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}v_{\overline{i,j}}\mathrm{d}w_{\overline{k,l}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{m,n}}}\Big), \tag{84}$$

$$\nabla^3_{w_{\overline{i,j}},w_{\overline{k,l}},w_{\overline{m,n}}} f = -\frac{2}{z^3}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{i,j}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{m,n}}} + \frac{1}{z^2}\Big(\frac{\mathrm{d}^2 z}{\mathrm{d}w_{\overline{i,j}}\mathrm{d}w_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{k,l}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}w_{\overline{k,l}}\mathrm{d}w_{\overline{m,n}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{i,j}}} + \frac{\mathrm{d}^2 z}{\mathrm{d}w_{\overline{i,j}}\mathrm{d}w_{\overline{k,l}}}\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{m,n}}}\Big) - \tag{85}$$

$$\frac{1}{z}\frac{\mathrm{d}^3 z}{\mathrm{d}w_{\overline{i,j}}\mathrm{d}w_{\overline{k,l}}\mathrm{d}w_{\overline{m,n}}} - \mathrm{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(w^{-1}). \tag{86}$$

Let us describe the derivatives of z. Let G_V, G_W be matrices containing the eigenvectors of V and W as columns, and let $\Gamma^{[i],V}, \Gamma^{[i],W}$ be the ith divided difference matrices of $\log(V)$ and $\log(W)$. $\Gamma^{[1]}$ and $\Gamma^{[2]}$ are described by Faybusovich and Zhou [2019]. For a vector of eigenvalues g, $\Gamma^{[3]}$ is given by:

$$\Gamma^{[3]}_{i,j,k,l} = \begin{cases} \frac{1}{3g_i^3} & g_i = g_j = g_k = g_l, \\ \frac{\Gamma^{[2]}_{i,i,i} - \Gamma^{[2]}_{i,i,j}}{g_i - g_j} & g_i = g_k = g_l, \\ \frac{\Gamma^{[2]}_{i,i,j} - \Gamma^{[2]}_{i,j,k}}{g_i - g_k} & g_i = g_l, \\ \frac{\Gamma^{[2]}_{j,k,l} - \Gamma^{[2]}_{i,j,k}}{g_l - g_i} & \text{otherwise.} \end{cases} \tag{87}$$

Following Faybusovich and Zhou [2019], let $\tilde{W} = G_V' W G_V$. Then:

$$\frac{\mathrm{d}z}{\mathrm{d}v_{\overline{i,j}}} = \mathrm{vec}(G_V(\tilde{W} \circ \Gamma^{[1],V})G_V')_{\overline{i,j}}, \tag{88a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}w_{\overline{i,j}}} = -\,\mathrm{vec}(I + \log(W) - \log(V))_{\overline{i,j}}. \tag{88b}$$

Let $\delta \in \mathbb{R}^{\mathrm{sd}(d)}$ be an arbitrary direction and $\Delta = \mathrm{mat}(\delta)$. Also let $\tilde{\Delta}^V = G_V' \Delta G_V$ and $\tilde{\Delta}^W = G_W' \Delta G_W$. The second and third derivatives of z are the operators that satisfy:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}v^2}[\delta] = \mathrm{vec}\left(G_V\, \mathrm{symm}\left([(\Gamma^{[2],V}_{:,:,k} \circ \tilde{W})\tilde{\Delta}^V_{:,k}]_{k \in [\![d]\!]}\right)G_V'\right), \tag{89a}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}v\mathrm{d}w}[\delta] = \mathrm{vec}\left(G_V(\tilde{\Delta}^V \circ \Gamma^{[1],V})G_V'\right), \tag{89b}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}w^2}[\delta] = -\,\mathrm{vec}\left(G_W(\tilde{\Delta}^W \circ \Gamma^{[1],W})G_W'\right), \tag{89c}$$

$$\frac{\mathrm{d}^3 z}{\mathrm{d}v^3}[\delta,\delta] = \mathrm{vec}\left(G_V\left[\textstyle\sum_{k,l \in [\![d]\!]} \Gamma^{[3],V}_{i,j,k,l}(\tilde{\Delta}^V_{k,l}\tilde{\Delta}^V_{l,j}\tilde{W}_{i,k} + \tilde{\Delta}^V_{k,l}\tilde{\Delta}^V_{l,i}\tilde{W}_{j,k} + \tilde{\Delta}^V_{k,i}\tilde{\Delta}^V_{l,j}\tilde{W}_{k,l})\right]_{i,j \in [\![d]\!]} G_V'\right), \tag{89d}$$

$$\frac{\mathrm{d}^3 z}{\mathrm{d}v^2\mathrm{d}w}[\delta,\delta] = 2\,\mathrm{vec}\left(G_V\left[\textstyle\sum_{k \in [\![d]\!]} \tilde{\Delta}^V_{p,k}\tilde{\Delta}^V_{k,q}\Gamma^{[2],V}_{p,q,k}\right]_{p,q \in [\![d]\!]} G_V'\right), \tag{89e}$$

$$\frac{\mathrm{d}^3 z}{\mathrm{d}v \mathrm{d}w^2}[\delta, \delta] = 0, \tag{89f}$$

$$\frac{\mathrm{d}^3 z}{\mathrm{d}w^3}[\delta, \delta] = -2 \operatorname{vec}\left(G_W \left[\textstyle\sum_{k \in [\![d]\!]} \tilde{\Delta}^W_{p,k} \tilde{\Delta}^W_{k,q} \Gamma^{[2],W}_{p,q,k}\right]_{p,q \in [\![d]\!]} G'_W\right). \tag{89g}$$

(89a) comes from Faybusovich and Zhou [2019, Section 4.1.1], which gives a formula for the Hessian of $\operatorname{tr}(C \log(W))$ in terms of a sparse matrix they call S. The action of S can be interpreted (for real and complex cases) as symmetrized matrix multiplication. For each column (and then row) k of $\tilde{\Delta}$, we apply $\Gamma^{[2],V}_{:,:,k} \circ \tilde{W}$ and stack the results.

2.20 Polynomial weighted sum-of-squares cone

Given matrices $P_r \in \mathbb{R}^{d \times s_r}, \forall r \in [\![N]\!]$, which are derived from basis polynomials evaluated at d interpolation points as described by Papp and Yildiz [2019], the interpolant-basis polynomial weighted sum-of-squares (WSOS) cone and its dual cone are:

$$\mathcal{K}_{\mathrm{SOS}(P)} = \left\{w \in \mathbb{R}^d : \exists S_r \in \mathbb{S}^{s_r}_{\succeq}, \forall r \in [\![N]\!], w = \textstyle\sum_{r \in [\![N]\!]} \operatorname{diag}\left(P_r S_r P'_r\right)\right\}, \tag{90a}$$

$$\mathcal{K}^*_{\mathrm{SOS}(P)} = \left\{w \in \mathbb{R}^d : Z(r) \in \mathbb{S}^{s_r}_{\succeq}, \forall r \in [\![N]\!]\right\}, \tag{90b}$$

where $Z(r) = P'_r \operatorname{Diag}(w) P_r, \forall r \in [\![N]\!]$. For the LHSCB [Nesterov, 2000, Papp and Yildiz, 2019]:

$$f(w) = -\textstyle\sum_{r \in [\![N]\!]} \operatorname{logdet}\left(Z(r)\right) \tag{91}$$

of $\mathcal{K}^*_{\mathrm{SOS}(P)}$ (note this is the dual cone), $\nu = \sum_{r \in [\![N]\!]} s_r$, and $t = e$ is a feasible point if the interpolation points are contained in the polynomial domain defined by P (assumed by default, else the user can pass in an known initial point). Let $T(r) = P_r Z(r)^{-1} P'_r, \forall r \in [\![N]\!]$, then:

$$\nabla_{w_i} f = -\textstyle\sum_{r \in [\![N]\!]} T(r)_{i,i}, \tag{92a}$$

$$\nabla^2_{w_i, w_j} f = \textstyle\sum_{r \in [\![N]\!]} \left(T(r)_{i,j}\right)^2, \tag{92b}$$

$$\nabla^3_{w_i, w_j, w_k} f = -2 \textstyle\sum_{r \in [\![N]\!]} T(r)_{i,j} T(r)_{j,k} T(r)_{k,i}. \tag{92c}$$

2.21 Polynomial WSOS positive semidefinite cone

Given matrices $P_r \in \mathbb{R}^{d_2 \times s_r}, \forall r \in [\![N]\!]$ defined as previously for $\mathcal{K}_{\mathrm{SOS}(P)}$, and given a side dimension d_1 of a symmetric matrix of polynomials (all using the same interpolant basis, for simplicity), the WSOS positive semidefinite cone and its dual cone are:

$$\mathcal{K}_{\mathrm{matSOS}(P)} = \begin{Bmatrix} w \in \mathbb{R}^{\mathrm{sd}(d_1) d_2} : \exists S_r \in \mathbb{S}^{d_1 s_r}_{\succeq}, \forall r \in [\![N]\!], \\ w_{\overline{i,j}} = \rho(i,j) \sum_{r \in [\![N]\!]} \operatorname{diag}\left(P_r (S_r)_{i,j} P'_r\right), \forall i \in [\![d_1]\!], j \in [\![i]\!] \end{Bmatrix}, \tag{93a}$$

$$\mathcal{K}^*_{\mathrm{matSOS}(P)} = \left\{w \in \mathbb{R}^{\mathrm{sd}(d_1) d_2} : Z(r) \in \mathbb{S}^{d_1 s_r}_{\succeq}, \forall r \in [\![N]\!]\right\}, \tag{93b}$$

where $w_{\overline{i,j}} \in \mathbb{R}^{d_2}$ is the contiguous slice of w corresponding to the interpolant basis values for the polynomial in the (i,j)th position of the lower triangle of the polynomial matrix, $(S)_{i,j}$ is the (i,j)th block in a symmetric block matrix S with square blocks of equal dimensions, and:

$$Z(r) = \left[P'_r \operatorname{Diag}\left(w_{\overline{i,j}}\right) P_r\right]_{i,j \in [\![d_1]\!]}, \tag{94}$$

where $[g(w_{\overline{i,j}})]_{i,j \in [\![d_1]\!]}$ is the symmetric block matrix with $g(w_{\overline{i,j}})$ in the (i,j)th block. For the LHSCB [Kapelevich et al., 2021]:

$$f(w) = -\textstyle\sum_{r \in [\![N]\!]} \operatorname{logdet}\left(Z(r)\right) \tag{95}$$

of $\mathcal{K}^*_{\mathrm{matSOS}(P)}$, $\nu = d_1 \sum_{r \in [\![N]\!]} s_r$, and t given by the w satisfying:

$$w_{\overline{i,j},p} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{96}$$

is a feasible point if the interpolation points are contained in the polynomial domain defined by P. Let:

$$T(r,p,q) = \left[\left(P_r(Z(r)^{-1})_{i,j}P_r'\right)_{p,q}\right]_{i,j\in[\![d_1]\!]}, \tag{97}$$

then:

$$\nabla_{w_{\overline{i,j},p}} f = -\textstyle\sum_{r\in[\![N]\!]}\rho(i,j)T(r,p,p)_{i,j}, \tag{98a}$$

$$\nabla^2_{w_{\overline{i,j},p},w_{\overline{k,l},q}} f = \textstyle\sum_{r\in[\![N]\!]} \operatorname{skron}_{\overline{i,j},\overline{k,l}}(T(r,p,q)), \tag{98b}$$

$$\nabla^3_{w_{\overline{i,j},p},w_{\overline{k,l},q},w_{\overline{m,n},s}} f = -\textstyle\sum_{r\in[\![N]\!]} \operatorname{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(T(r,q,p),T(r,p,s),T(r,s,q)). \tag{98c}$$

2.22 Polynomial WSOS L1 norm cone

Given matrices $P_r \in \mathbb{R}^{d_2\times s_r}, \forall r\in[\![N]\!]$ defined as previously for $\mathcal{K}_{\mathrm{SOS}(P)}$, and given a side dimension d_1+1 of a vector of polynomials (all using the same interpolant basis, for simplicity), the WSOS ℓ_1 norm cone and its dual cone are:

$$\mathcal{K}_{\ell_1\,\mathrm{SOS}(P)} = \left\{w\in\mathbb{R}^{d_1d_2} : \exists v_2,\dots,v_{d_1}\in\mathbb{R}^{d_2}, (v_i,w_i)\in\mathcal{K}_{\mathrm{SOS}\,\ell_2(P)}, \forall i\in 2,\dots,d, w_1 = \textstyle\sum_{i\in 2,\dots,d} v_i\right\}, \tag{99a}$$

$$\mathcal{K}^*_{\ell_1\,\mathrm{SOS}(P)} = \left\{w\in\mathbb{R}^{d_1d_2} : (w_1,w_i)\in\mathcal{K}^*_{\mathrm{SOS}\,\ell_2(P)}, \forall i\in 2,\dots,d\right\}, \tag{99b}$$

where $w_i\in\mathbb{R}^{d_2}$ is the contiguous slice of w corresponding to the interpolant basis values for the polynomial in the ith position the polynomial vector, and:

$$Z(r,k) = \begin{bmatrix} P_r'\operatorname{diag}(w_1)P_r & P_r'\operatorname{diag}(w_k)P_r \\ P_r'\operatorname{diag}(w_k)P_r & P_r'\operatorname{diag}(w_1)P_r \end{bmatrix}, \quad \forall k\in 2,\dots,d_1. \tag{100}$$

For the LHSCB [Kapelevich et al., 2021]:

$$f(w) = \textstyle\sum_{r\in[\![N]\!]}\sum_{k\in 2,\dots,d_1} -\operatorname{logdet} Z(r,k)_{1,1} - Z(r,k)_{1,2}Z(r,k)^{-1}_{1,1}Z(r,k)_{1,2} - \operatorname{logdet}\left(P_r'\operatorname{diag}(w_1)P_r\right) \tag{101}$$

$$= \textstyle\sum_{r\in[\![N]\!]}\sum_{k\in 2,\dots,d_1} -\operatorname{logdet} Z(r,k) + (d_1-2)\operatorname{logdet}\left(P_r'\operatorname{diag}(w_1)P_r\right) \tag{102}$$

of $\mathcal{K}^*_{\ell_1\,\mathrm{SOS}(P)}$, $\nu = m\sum_{r\in[\![N]\!]}s_r$ and t given by w satisfying:

$$w_{i,p} = \begin{cases} 1, & i=1, \\ 0, & \text{otherwise}, \end{cases} \tag{103}$$

is a feasible point if the interpolation points are contained in the polynomial domain defined by P. Let:

$$Q(r,k,i) = \left(P_rZ(r,k)P_r'\right)_{1,i}, \qquad i\in[\![2]\!], k\in 2,\dots,d_1. \tag{104}$$

The gradient and Hessian of the barrier are:

$$\nabla_{w_{i,p}} f = \begin{cases} -2\sum_{k\in 2,\dots,d_1} Q(r,k,1)_{p,p} + (d_1-2)\left(P_r\left(P_r'\operatorname{diag}(w_1)P_r\right)^{-1}P_r'\right)_{p,p}, & i=1, \\ -2Q(r,i,2)_{p,p}, & i\neq 1, \end{cases} \tag{105}$$

$$\nabla_{w_{i,p},w_{j,q}} f = \begin{cases} 2\sum_{k\in 2,\dots,d_1}\sum_{l\in\{1,2\}}\left(Q(r,k,l)_{p,q}\right)^2 - (d_1-2)\left(\left(P_r\left(P_r'\operatorname{diag}(w_1)P_r\right)^{-1}P_r'\right)_{p,q}\right)^2, & i=j=1, \\ 4Q(r,i,1)_{p,q}R(r,i,2)_{p,q}, & i=1 \text{ or } j=1 \\ 2\sum_{k\in\{1,2\}}\left(Q(r,i,k)_{p,q}\right)^2, & i=j\neq 1 \\ 0, & \text{otherwise}. \end{cases} \tag{106}$$

2.23 Polynomial WSOS L2 norm cone

Given matrices $P_r \in \mathbb{R}^{d_2 \times s_r}, \forall r \in [\![N]\!]$ defined as previously for $\mathcal{K}_{\text{SOS}(P)}$, and given a side dimension d_1 of a vector of polynomials (all using the same interpolant basis, for simplicity), the WSOS ℓ_2 norm cone and its dual cone are:

$$\mathcal{K}_{\ell_2\,\text{SOS}(P)} = \left\{ \begin{matrix} w \in \mathbb{R}^{d_1 d_2} : \exists S_r \in \mathbb{S}_{\succeq}^{d_1 s_r}, \forall r \in [\![N]\!], w_1 = \sum_{r \in [\![N]\!]} \sum_{i \in [\![d_1]\!]} \operatorname{diag}\left(P_r (S_r)_{i,i} P_r'\right) \\ w_i = \sum_{r \in [\![N]\!]} \operatorname{diag}\left(P_r \left((S_r)_{i,1} + (S_r)_{i,1}'\right) P_r'\right), \forall i \in 2, \dots, d_1 \end{matrix} \right\}, \tag{107a}$$

$$\mathcal{K}^*_{\ell_2\,\text{SOS}(P)} = \left\{ w \in \mathbb{R}^{d_1 d_2} : Z(r)_{1,1} - \textstyle\sum_{i \in 2,\dots,d_1} Z(r)_{1,i} Z(r)_{1,1}^{-1} Z(r)_{i,1} \in \mathbb{S}_{\succeq}^{d_1 s_r}, \forall r \in [\![N]\!] \right\}, \tag{107b}$$

where $w_i \in \mathbb{R}^{d_2}$ is the contiguous slice of w corresponding to the interpolant basis values for the polynomial in the ith position the polynomial vector, $(S)_{i,j}$ is the (i,j)th block in a symmetric block matrix S with square blocks of equal dimensions, and:

$$Z(r)_{i,j} = \begin{cases} P_r' \operatorname{Diag}(w_1) P_r, & i = j, \\ P_r' \operatorname{Diag}(w_j) P_r, & i = 1, j \neq 1, \\ P_r' \operatorname{Diag}(w_i) P_r, & i \neq 1, j = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{108}$$

For the LHSCB [Kapelevich et al., 2021]:

$$f(w) = \textstyle\sum_{r \in [\![N]\!]} \left(-\operatorname{logdet}(Z(r)) + (d_1 - 2) \operatorname{logdet}\left(P_r' \operatorname{diag}(w_1) P_r\right)\right) \tag{109}$$

of $\mathcal{K}^*_{\ell_2\,\text{SOS}(P)}$, $\nu = 2 \sum_{r \in [\![N]\!]} s_r$ and t given by w satisfying:

$$w_{i,p} = \begin{cases} 1, & i = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{110}$$

is a feasible point if the interpolation points are contained in the polynomial domain defined by P. Let:

$$R(r,i,j) = \left(P_r \left(Z(r)^{-1}\right)_{i,j} P_r'\right). \tag{111}$$

$$\nabla_{w_{i,p}} f = \begin{cases} -\sum_{r \in [\![N]\!]} \sum_{j \in [\![1..m]\!]} R(r,j,j)_{p,p} + (d_1 - 2)\left(P_r \left(P_r' \operatorname{diag}(w_1) P_r\right)^{-1} P_r'\right)_{p,p}, & i = 1, \\ -2 \sum_{r \in [\![N]\!]} R(r,i,1)_{p,p}, & i \neq 1, \end{cases} \tag{112a}$$

$$\nabla_{w_{i,p}, w_{j,q}} f = \begin{cases} \sum_{r \in [\![N]\!]} \sum_{k \in [\![d_1]\!]} \sum_{l \in [\![d_1]\!]} \left(R(r,k,l)_{u,u'}\right)^2 - (d_1 - 2)\left(\left(P_r \left(P_r' \operatorname{diag}(w_1) P_r\right)^{-1} P_r'\right)_{p,q}\right)^2, & i = j = 1, \\ 2 \sum_{r \in [\![N]\!]} \sum_{k \in [\![d_1]\!]} R(r,k,1)_{p,q} R(r,k,j)_{p,q}, & i = 1, j \neq 1, \\ 2 \sum_{r \in [\![N]\!]} \sum_{k \in [\![d_1]\!]} R(r,1,k)_{p,q} R(r,i,k)_{p,q}, & i \neq 1, j = 1, \\ 2 \sum_{r \in [\![N]\!]} \left(R(r,1,1)_{p,q} R(r,i,j)_{p,q} + R(r,i,1)_{p,q} R(r,1,j)_{p,q}\right), & i \neq 1, j \neq 1. \end{cases} \tag{113a}$$

A Matrix operators

We define several operators that are convenient for expressing derivatives of barrier functions for matrix cones. For brevity, we overload the operators below if all inputs are the same, e.g. $\operatorname{skron}_{\overline{i,j},\overline{k,l}}(X) = \operatorname{skron}_{\overline{i,j},\overline{k,l}}(X,X)$. We define the symmetrizing operator $\operatorname{symm} : \mathbb{R}^{d \times d} \to \mathbb{S}^{\operatorname{sd}(d)}$ as:

$$\operatorname{symm}(X) = X + X'. \tag{114}$$

A.1 Two-input Kronecker-like products

If $M \in \mathbb{R}^{d_1 \times d_2}$ is a Kronecker-like product of two matrices, and $U \in \mathbb{R}^{d_3 \times d_4}$ and $V \in \mathbb{R}^{d_5 \times d_6}$ are matrices such that $\overline{d_3, d_4} = d_1$ and $\overline{d_5, d_6} = d_2$, then $U \cdot M$ and $M \cdot V$ are the matrices:

$$U \cdot M = \mathrm{mat}(M' \,\mathrm{vec}(U)), \tag{115a}$$

$$M \cdot V = \mathrm{mat}(M \,\mathrm{vec}(V)). \tag{115b}$$

A.1.1 Symmetric preimage

Let $X, Y \in \mathbb{R}^{d \times d}$, $U \in \mathbb{S}^d$. The symmetric preimage $\mathrm{skron} : \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \to \mathbb{S}^{\mathrm{sd}(d)}$ (c.f. Alizadeh et al. [1998, Appendix]) satisfies:

$$\mathrm{skron}(X, Y) \cdot U = \tfrac{1}{2} \,\mathrm{symm}(XUY'). \tag{116}$$

The elementwise definition is:

$$\mathrm{skron}_{\overline{i,j},\overline{k,l}}(X, Y) = \tfrac{1}{4} \rho(i, j) \rho(k, l)(X_{i,k} Y_{j,l} + X_{i,l} Y_{j,k} + X_{j,l} Y_{i,k} + X_{j,k} Y_{i,l}). \tag{117}$$

A.1.2 General preimage

Let $X \in \mathbb{R}^{d_1 \times d_1}$, $Y \in \mathbb{R}^{d_2 \times d_2}$, $U \in \mathbb{R}^{d_1 \times d_2}$. The general preimage $\mathrm{gkron} : \mathbb{R}^{d_1 \times d_1} \times \mathbb{R}^{d_2 \times d_2} \to \mathbb{R}^{d_1 d_2 \times d_1 d_2}$ satisfies:

$$\mathrm{gkron}(X, Y) \cdot U = XUY'. \tag{118}$$

The elementwise definition is:

$$\mathrm{gkron}_{\overline{i,j},\overline{k,l}}(X, Y) = X_{i,k} Y_{j,l}. \tag{119}$$

A.1.3 Adjoint preimage

Let $X, Y, U \in \mathbb{R}^{d_1 \times d_2}$. The adjoint preimage $\mathrm{akron} : \mathbb{R}^{d_1 \times d_2} \times \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^{d_1 d_2 \times d_1 d_2}$ satisfies:

$$\mathrm{akron}(X, Y) \cdot U = XU'Y + YU'X. \tag{120}$$

The elementwise definition is:

$$\mathrm{akron}_{\overline{i,j},\overline{k,l}}(X, Y) = X_{i,l} Y_{k,j} + X_{k,j} Y_{i,l}. \tag{121}$$

A.1.4 Mixed preimage

Let $X, U \in \mathbb{S}^{d_1}$, $Y, V \in \mathbb{R}^{d_1 \times d_2}$. The output of the mixed preimage $\mathrm{mkron} : \mathbb{S}^{d_1} \times \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^{\mathrm{sd}(d_1) \times d_1 d_2}$ is not square, and we arbitrarily choose the number of rows to equal the dimension of the first input. The operator mkron satisfies:

$$U \cdot \mathrm{mkron}(X, Y) = 2XUY, \tag{122a}$$

$$\mathrm{mkron}(X, Y) \cdot V = \mathrm{symm}(XVY'). \tag{122b}$$

The elementwise definition is:

$$\mathrm{mkron}_{\overline{i,j},\overline{k,l}}(X, Y) = \rho(i, j)(X_{i,k} Y_{j,l} + X_{j,k} Y_{i,l}). \tag{123}$$

A.2 Three-input Kronecker-like products

If $M \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ is a Kronecker-like product of three matrices, let $\mathrm{mat}_{d_4,d_5}(M) \in \mathbb{R}^{d_4 \times d_5}$ be a *reshaped* representation of M obtained by stacking columns so that $d_1 d_2 d_3 = d_4 d_5$. Let $U \in \mathbb{R}^{d_4 \times d_5}$ and $V \in \mathbb{R}^{d_6 \times d_7}$ be matrices such that $d_4 d_5 = d_1$ and $d_6 d_7 = d_3$, then $U \cdot M$ and $M \cdot V$ are the matrices:

$$U \cdot M = \mathrm{mat}_{d_2,d_3}(\mathrm{mat}_{d_1,d_2 d_3}(M)' \,\mathrm{vec}(U)), \tag{124a}$$

$$M \cdot V = \mathrm{mat}_{d_1,d_2}(\mathrm{mat}_{d_1 d_2,d_3}(M) \,\mathrm{vec}(V)). \tag{124b}$$

A.2.1 Symmetric preimage

Let $X, Y, Z \in \mathbb{R}^{d\times d}$, $U, V \in \mathbb{S}^d$. The symmetric preimage skron : $\mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}^{\mathrm{sd}(d)\times\mathrm{sd}(d)\times\mathrm{sd}(d)}$ satisfies:

$$(\mathrm{skron}(X, Y, Z) \cdot U) \cdot V = \mathrm{symm}(XUYVZ). \tag{125}$$

The elementwise definition is:

$$\begin{aligned}\mathrm{skron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(X, Y, Z) = \tfrac{1}{4}\rho(i, j)\rho(k, l)\rho(m, n)(&X_{i,n}Y_{k,j}Z_{m,l} + X_{j,n}Y_{k,i}Z_{m,l} + X_{i,n}Y_{l,j}Z_{m,k} +\\ &X_{j,n}Y_{l,i}Z_{m,k} + X_{i,m}Y_{k,j}Z_{n,l} + X_{j,m}Y_{k,i}Z_{n,l} + X_{i,m}Y_{l,j}Z_{n,k} + X_{j,m}Y_{l,i}Z_{n,k}).\end{aligned} \tag{126}$$

A.2.2 General preimage

Let $X \in \mathbb{R}^{d_1\times d_1}$, $Z \in \mathbb{R}^{d_2\times d_2}$, $Y, U, V \in \mathbb{R}^{d_1\times d_2}$. The general preimage gkron : $\mathbb{R}^{d_1\times d_1} \times \mathbb{R}^{d_1\times d_2} \times \mathbb{R}^{d_2\times d_2} \to \mathbb{R}^{d_1 d_2\times d_1 d_2\times d_1 d_2}$ satisfies:

$$(\mathrm{gkron}(X, Y, Z) \cdot U) \cdot V = \mathrm{gkron}(X, Z) \cdot (\mathrm{akron}(U, V) \cdot Y) + X\,\mathrm{symm}(UZ'V')Y + Y\,\mathrm{symm}(U'X'V)Z. \tag{127}$$

The elementwise definition is:

$$\begin{aligned}\mathrm{gkron}_{\overline{i,j},\overline{k,l}}(X, Y, Z) = &X_{i,m}Y_{k,n}Z_{j,l} + X_{k,m}Y_{i,n}Z_{j,l} + X_{i,k}Y_{m,j}Z_{l,n} +\\ &X_{i,k}Y_{m,l}Z_{j,n} + X_{i,m}Y_{k,j}Z_{l,n} + X_{k,m}Y_{i,l}Z_{j,n}.\end{aligned} \tag{128}$$

A.2.3 Adjoint preimage

Let $X, Y, Z, U, V \in \mathbb{R}^{d_1\times d_2}$. The adjoint preimage akron : $\mathbb{R}^{d_1\times d_2} \times \mathbb{R}^{d_1\times d_2} \times \mathbb{R}^{d_1\times d_2} \to \mathbb{R}^{d_1 d_2\times d_1 d_2\times d_1 d_2}$ satisfies:

$$\begin{aligned}(\mathrm{akron}(X, Y, Z) \cdot U) \cdot V = &\mathrm{akron}(X, Y) \cdot (\mathrm{akron}(U, V) \cdot Z) + \mathrm{akron}(X, Z) \cdot (\mathrm{akron}(U, V) \cdot Y) +\\ &\mathrm{akron}(Y, Z) \cdot (\mathrm{akron}(U, V) \cdot X).\end{aligned} \tag{129}$$

The elementwise definition is:

$$\begin{aligned}\mathrm{akron}_{\overline{i,j},\overline{k,l}}(X, Y, Z) = &X_{i,l}Y_{m,j}Z_{k,n} + X_{k,j}Y_{m,l}Z_{i,n} + X_{i,l}Z_{m,j}Y_{k,n} + X_{k,j}Z_{m,l}Y_{i,n} +\\ &Y_{i,l}Z_{m,j}X_{k,n} + Y_{k,j}Z_{m,l}X_{i,n} + Y_{i,l}X_{m,j}Z_{k,n} + Y_{k,j}X_{m,l}Z_{i,n} +\\ &Z_{i,l}X_{m,j}Y_{k,n} + Z_{k,j}X_{m,l}Y_{i,n} + Z_{i,l}Y_{m,j}X_{k,n} + Z_{k,j}Y_{m,l}X_{i,n}.\end{aligned} \tag{130}$$

A.2.4 Mixed preimage

Let $U_1, U_2 \in \mathbb{S}^{d_1}$, $V_1, V_2 \in \mathbb{R}^{d_1\times d_2}$. We define the mixed preimage mkron for three different cases of input combinations.

Case 1: Let $X, Z \in \mathbb{S}^{d_1}$, $Y \in \mathbb{R}^{d_1\times d_2}$, then mkron : $\mathbb{S}^{d_1} \times \mathbb{R}^{d_1\times d_2} \times \mathbb{S}^{d_1} \to \mathbb{R}^{\mathrm{sd}(d_1)\times\mathrm{sd}(d_1)\times d_1 d_2}$ satisfies:

$$U_2 \cdot (U_1 \cdot \mathrm{mkron}(X, Y, Z)) = 4(\mathrm{mkron}(X, Y) \cdot (\mathrm{skron}(U_1, U_2) \cdot Z) + \mathrm{mkron}(Z, Y) \cdot (\mathrm{skron}(U_1, U_2) \cdot X)), \tag{131a}$$

$$(U_1 \cdot \mathrm{mkron}(X, Y, Z)) \cdot V_1 = 2\,\mathrm{symm}(XU_1\,\mathrm{symm}(ZV_1Y') + \mathrm{symm}(XV_1Y')U_1Z). \tag{131b}$$

The elementwise definition is:

$$\begin{aligned}\mathrm{mkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(X, Y, Z) = \rho(i, j)\rho(k, l)(&X_{i,k}Y_{l,n}Z_{j,m} + X_{i,k}Y_{j,n}Z_{l,m} + X_{j,l}Y_{i,n}Z_{k,m} + X_{j,l}Y_{k,n}Z_{i,m} +\\ &X_{i,l}Y_{k,n}Z_{j,m} + X_{i,l}Y_{j,n}Z_{k,m} + X_{j,k}Y_{i,n}Z_{l,m} + X_{j,k}Y_{l,n}Z_{i,m} +\\ &X_{j,m}Y_{l,n}Z_{i,k} + X_{l,m}Y_{j,n}Z_{i,k} + X_{k,m}Y_{i,n}Z_{j,l} + X_{i,m}Y_{k,n}Z_{j,l} +\\ &X_{j,m}Y_{k,n}Z_{i,l} + X_{k,m}Y_{j,n}Z_{i,l} + X_{l,m}Y_{i,n}Z_{j,k} + X_{i,m}Y_{l,n}Z_{j,k}).\end{aligned} \tag{132}$$

Case 2: Let $X \in \mathbb{S}^{d_1}$, $Y, Z \in \mathbb{R}^{d_1 \times d_2}$, then mkron : $\mathbb{S}^{d_1} \times \mathbb{R}^{d_1 \times d_2} \times \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^{\mathrm{sd}(d_1) \times d_1 d_2 \times d_1 d_2}$ satisfies:

$$(\mathrm{mkron}(X,Y,Z) \cdot V_1) \cdot V_2 = \mathrm{symm}(XV_2(\mathrm{akron}(Y',Z') \cdot V_1') + YV_2' \,\mathrm{symm}(XV_1Z') + ZV_2' \,\mathrm{symm}(XV_1Y')), \tag{133a}$$

$$(U_1 \cdot \mathrm{mkron}(X,Y,Z)) \cdot V_1 = 2(X \,\mathrm{symm}(U_1ZV_1')Y + X \,\mathrm{symm}(U_1YV_1')Z + \mathrm{akron}(Y,Z) \cdot (U_1XV_1)). \tag{133b}$$

The elementwise definition is:

$$\begin{aligned}\mathrm{mkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(X,Y,Z) = \rho(i,j)(&X_{i,m}Z_{k,n}Y_{j,l} + X_{k,m}Z_{i,n}Y_{j,l} + X_{i,k}Z_{m,l}Y_{j,n} + X_{j,k}Z_{m,l}Y_{i,n} + \\ &X_{k,m}Z_{i,l}Y_{j,n} + X_{j,m}Z_{i,l}Y_{k,n} + X_{i,m}Y_{k,n}Z_{j,l} + X_{k,m}Y_{i,n}Z_{j,l} + \\ &X_{i,k}Y_{m,l}Z_{j,n} + X_{j,k}Y_{m,l}Z_{i,n} + X_{k,m}Y_{i,l}Z_{j,n} + X_{j,m}Y_{i,l}Z_{k,n}).\end{aligned} \tag{134}$$

Case 3: Let $X, Y \in \mathbb{S}^{d_1}$, $Z \in \mathbb{S}^{d_2}$, then mkron : $\mathbb{S}^{d_1} \times \mathbb{S}^{d_1} \times \mathbb{S}^{d_2} \to \mathbb{R}^{\mathrm{sd}(d_1) \times d_1 d_2 \times d_1 d_2}$ satisfies:

$$(\mathrm{mkron}(X,Y,Z) \cdot V_1) \cdot V_2 = \mathrm{symm}(X \,\mathrm{symm}(V_1ZV_2')Y), \tag{135a}$$

$$(U_1 \cdot \mathrm{mkron}(X,Y,Z)) \cdot V_1 = 2 \,\mathrm{symm}(XU_1Y)V_1Z. \tag{135b}$$

The elementwise definition is:

$$\mathrm{mkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(X,Y,Z) = \rho(i,j)Z_{l,n}(X_{i,k}Y_{j,m} + X_{j,k}Y_{i,m} + X_{i,m}Y_{j,k} + X_{j,m}Y_{i,k}). \tag{136}$$

A.3 Dot product-like operators

Let $X \in \mathbb{S}^d$, $x = \mathrm{vec}(X)$. We define:

$$\mathrm{sdot}_{\overline{i,j},\overline{k,l}}(X) = x_{\overline{i,j}}x_{\overline{k,l}}, \tag{137}$$

$$\mathrm{sdot}_{\overline{i,j},\overline{k,l},\overline{m,n}}(X) = x_{\overline{i,j}}x_{\overline{k,l}}x_{\overline{m,n}}, \tag{138}$$

$$\mathrm{sdotkron}_{\overline{i,j},\overline{k,l},\overline{m,n}}(X) = x_{\overline{i,j}} \,\mathrm{skron}_{\overline{k,l},\overline{m,n}}(X) + x_{\overline{k,l}} \,\mathrm{skron}_{\overline{i,j},\overline{m,n}}(X) + x_{\overline{m,n}} \,\mathrm{skron}_{\overline{i,j},\overline{k,l}}(X). \tag{139}$$

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